A FINITE ELEMENT ANALYSIS IN BALANCED NORMS FOR A COUPLED SYSTEM OF SINGULARLY PERTURBED REACTION-DIFFUSION EQUATIONS

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Abstract. In this paper we study approximations of a singularly perturbed system of two coupled reaction-diffusion equations, in one dimension, by using finite elements on graded meshes. When the parameters are of different magnitudes, the solution exhibits in general two distinct but overlapping boundary layers. We prove that, when the mesh grading parameter is appropriately chosen, optimal error estimations in a balanced norm for piecewise linear elements can be obtained. Supporting numerical results are also presented.

1. Introduction

Singularly perturbed systems of ordinary differential equations appear when investigating diffusion processes complicated by chemical reactions where the parameters multiplying the highest derivatives characterize the diffusion coefficient of the substances [24]. Another interesting application is in ecology, where reaction-diffusion systems can be used to describe the prey-predator interaction species [8]. There are several papers devoted to the numerical approximations of singularly perturbed systems of coupled reaction-diffusion equations (see, for example, [5, 11, 13, 14, 15, 17, 19, 22]). Problems with different layers in one coordinate direction or systems of reaction-diffusion equations, still present several challenges. In [14], the authors consider a system of two reaction-diffusion equations in one dimension, with different small parameters multiplying the second-order derivatives in the equations. In that work, they analyze finite element approximations, with Shishkin and Bakhvalov meshes, and obtain error estimates in the energy norm.

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It is well known that, for singularly perturbed reaction-diffusion problems, the natural energy norm is not balanced and so, the interest of work with balanced norms, which reflect the behavior of layers more accurately in the finite element method, have increased in the last years (see [1, 2, 3, 10, 18, 20, 23] and the references therein). The error estimation using balanced norms for systems of two coupled reaction-diffusion equations with different small parameters, still presents open questions even in the one dimensional case such, for instance, how to design an efficient mesh according to different perturbed parameters [12, 21, 22].

In this paper, we analyze the finite element approximation of a singularly perturbed system of ordinary differential equations, by using piecewise linear elements on graded meshes. First, we analyze the general case of variable coefficients but with the same small perturbation parameter in both equations. Then, we consider the case of different small parameters multiplying the second-order derivatives but assuming constant coefficients in both equations.

Graded meshes satisfy some interesting properties. One of the most relevant is the fact that a mesh designed for some value of the perturbation parameter also works well also for larger values of it and we can obtain optimal error estimates by using graded meshes (of the same type to those introduced in [3]), designed according to the smallest parameter of the system.

To achieve these optimal error estimates, our analysis requires the introduction of appropriate L^2-projections and the analysis of their stability and interpolation capabilities on graded meshes.

The rest of the paper is organized as follows. In Section 2, we state the reaction-diffusion coupled problem and recall a priori estimates for the solution. In Section 3, we introduce the graded meshes which we use for the finite element discretization. We also obtain interpolation error estimates, stability results for L^2-projections and a preliminary result about the numerical approximation of one singularly perturbed reaction-diffusion equation. Section 4 contains our main results concerning the optimal approximation error estimates in balanced norms. In Section 5, we present some numerical examples which show the good performance of the proposed approach. Finally, we end the paper drawing some conclusions in Section 6.

2. Problem Statement

We consider the following system of two coupled reaction-diffusion equations in $I = (0, 1)$:

$$\begin{cases} -\varepsilon^2 u_1''(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x) \\ -\mu^2 u_2''(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x) \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0 \end{cases} \tag{1}$$

where $\mathbf{f} = (f_1(x), f_2(x))$ and $A = (a_{ij}(x))_{1\le i,j\le 2}$ are smooth on $[0,1]$.

We are interested in the singularly perturbed case, that is, when at least one of the singular parameters ε or μ is $\ll 1$. Due to that, the solution $\mathbf{u} = (u_1, u_2)^T$ may exhibit boundary layers of width $O(\varepsilon \ln \varepsilon)$ and $O(\mu \ln \mu)$ at $x = 0$ and $x = 1$, which could overlap and interact according to the relative size of ε and μ (see, for example, [17, 21]).

From now on, we assume

$$0 < \epsilon \le \mu \le 1.$$

The matrix A has bounded entries $a_{ij}(x)$ and we assume that $a_{ii} > 0, a_{ij} \le 0, i \ne j$, $1 \le i, j \le 2$, and there exists $\alpha \ne 0$ such that

$$\min_{[0,1]}\{a_{11} + a_{12}, a_{21} + a_{22}\} \ge \alpha^2. \tag{2}$$

Since, as a consequence of (2), A is an M–matrix, from [15, Theorem 2.2 and Remark 2.5], we can also assume that

$$\xi^t A \xi \ge \alpha^2 \xi^t \xi, \quad \forall \xi \in \mathbb{R}^2. \tag{3}$$

We denote with boldface the spaces consisting of vector valued functions. The norms and seminorms in $H^m(\mathcal{D})$ and $\mathbf{H}^m(\mathcal{D})$, with m an integer, are denoted by $\|\cdot\|_{m,\mathcal{D}}$ and $|\cdot|_{m,\mathcal{D}}$ respectively and $(\cdot,\cdot)_{\mathcal{D}}$ denotes the inner product in $L^2(\mathcal{D})$ or $\mathbf{L}^2(\mathcal{D})$ for any subdomain $\mathcal{D} \subset I$. The domain subscript is dropped for the case $\mathcal{D} = I$. We also denote by $\langle\cdot,\cdot\rangle$ the Euclidean product on $\mathbb{R}^d$ and $|\cdot|^2 = \langle\cdot,\cdot\rangle$.

Let

$$\mathbf{B}(\mathbf{u},\mathbf{v}) := \int_0^1 \varepsilon^2 u_1' v_1' + a_{11}u_1v_1 + a_{12}u_2v_1 + \mu^2 u_2' v_2' + a_{21}u_1v_2 + a_{22}u_2v_2,$$

and

$$\mathbf{L}(\mathbf{v}) := \int_0^1 f_1 v_1 + f_2 v_2.$$

The variational formulation is: find $\mathbf{u} = (u_1, u_2)^T \in \mathbf{V} := \mathbf{H}_0^1(I)$ such that

$$\mathbf{B}(\mathbf{u},\mathbf{v}) = \mathbf{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}.$$

The following result, which is a consequence of [14, Lemma 1] and [17, Lemma 4], shows the behaviour of the exact solution to be approximated and its derivatives up to order 2. Similar estimates for all the derivatives, assuming analytic data, can be found in [21].

Lemma 2.1. *Let* $\mathbf{u}$ *be the solution to* (1)*. Then there exists a constant* C*, such that for all* $x \in [0,1]$ *we have*

$$\mathbf{u} = \mathbf{v} + \mathbf{w},$$

where the regular solution component $\mathbf{v}$ *satisfies*

$$|v_i'| \leq C \quad \text{ and } \quad |v_i''| \leq C, \quad i = 1, 2$$

while the layer component $\mathbf{w}$ *satisfy*

$$\begin{aligned}
|w_1'| &\leq C\left(\varepsilon^{-1}e^{-\alpha\frac{x}{\varepsilon}} + \mu^{-1}e^{-\alpha\frac{x}{\mu}} + \varepsilon^{-1}e^{-\alpha\frac{1-x}{\varepsilon}} + \mu^{-1}e^{-\alpha\frac{1-x}{\mu}}\right),\\
|w_2'| &\leq C\left(\mu^{-1}e^{-\alpha\frac{x}{\mu}} + \mu^{-1}e^{-\alpha\frac{1-x}{\mu}}\right),\\
|w_1''| &\leq C\left(\varepsilon^{-2}e^{-\alpha\frac{x}{\varepsilon}} + \mu^{-2}e^{-\alpha\frac{x}{\mu}} + \varepsilon^{-2}e^{-\alpha\frac{1-x}{\varepsilon}} + \mu^{-2}e^{-\alpha\frac{1-x}{\mu}}\right),\\
|w_2''| &\leq C\left(\mu^{-2}e^{-\alpha\frac{x}{\mu}} + \mu^{-2}e^{-\alpha\frac{1-x}{\mu}}\right).
\end{aligned}$$

Given a partition $\mathcal{T}_h = \{0 = x_0 < x_1 < \ldots < x_M = 1\}$, we denote $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$ with $1 \leq i \leq M$, and we define

$$\begin{aligned}
\hat{h}_0 &= h_1,\\
\hat{h}_k &= \frac{1}{2}(h_{k+1} + h_k), \quad 1 \leq k \leq M-1,\\
\hat{h}_M &= h_M.
\end{aligned}$$

We consider the finite element space

$$\mathbf{V}_h = \left\{\mathbf{v} = (v_1, v_2)^T \in \mathbf{V} : v_j|_{I_i} \in P_1(I_i), i = 1, \ldots, M, j = 1, 2\right\},$$

and the space $V_h = \{v \in H_0^1(I) : v|_{I_i} \in P_1(I_i), i = 1, \ldots, M\}$, where $P_1(D)$ denotes the space of linear polynomials on a domain D.

We denote by $\phi_i, i = 0, \ldots, M$, the classical Lagrange linear basis functions such that $\phi_i(x_j) = \delta_{ij}, i, j = 0, \ldots, M$.

For a generic interval $I_\ell = (x_{\ell-1}, x_\ell)$ of the partition, we denote $x_1^\ell = x_{\ell-1}$ and $x_2^\ell = x_\ell$. We also set the local basis functions $\phi_1^\ell = \phi_{\ell-1}$ and $\phi_2^\ell = \phi_\ell$, and the local lengths $\hat{h}_1^\ell = \hat{h}_{\ell-1}$ and $\hat{h}_2^\ell = \hat{h}_\ell$.

The conforming finite element formulation is given by: find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\mathbf{B}(\mathbf{u}_h, \mathbf{v}) = \mathbf{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h.$$

Using the classical theory we can affirm that the problem is well-defined and there exists a unique solution $\mathbf{u}_h \in \mathbf{V}_h$. Error estimates, which are robust in the natural energy norm

$$\|\mathbf{u}\|_e^2 = \varepsilon^2\|u_1'\|_0^2 + \mu^2\|u_2'\|_0^2 + \alpha^2(\|u_1\|_0^2 + \|u_2\|_0^2),$$

were obtained for different kind of meshes (see, for example, [5, 14, 19]). In the present article, we are interested in obtaining robust error estimates in a balanced norm. To this end, following [22], we consider the balanced norm which is defined by introducing a different scaling of the H^1 seminorm:

$$|||\mathbf{u}|||^2 = \varepsilon\|u_1'\|_0^2 + \mu\|u_2'\|_0^2 + \alpha^2(\|u_1\|_0^2 + \|u_2\|_0^2)$$

As explained in [22], this norm reflects the layer behavior correctly.

3. Graded meshes and preliminary results

In this section we introduce the graded meshes that we use for the finite element approximation of problem (1). We obtain interpolation error estimates, stability results for L^2-projections and also a preliminary result about the numerical approximation of one singularly perturbed reaction-diffusion equation.

3.1. **Graded meshes.**

Let us introduce a family of graded meshes $\mathcal{T}_h$ as in [3]. Let be $h > 0$, related to the mesh size, and let

$$\gamma = 1 - \frac{1}{2\log\frac{1}{\varepsilon}} \quad \text{and} \quad s = \frac{1}{1-\gamma}, \tag{4}$$

be the grading parameters. Then, the graded meshes are obtained in the following way.

Let $x_0, x_1, \dots, x_{\text{mid}}$ be the grid points on the interval $[0, \frac{1}{2}]$ given by

$$\begin{cases} x_0 = 0, \\ x_1 = h^s, \\ x_{i+1} = x_i + hx_i^\gamma, \quad i = 1, \dots, \text{mid}-2, \\ x_{\text{mid}} = \dfrac{1}{2}, \end{cases} \tag{5}$$

where mid is such that $x_{\text{mid}-1} < \frac{1}{2}$ and $x_{\text{mid}-1} + hx_{\text{mid}-1}^\gamma \geq \frac{1}{2}$. We assume that the interval $(x_{\text{mid}-1}, x_{\text{mid}})$ is not too small in comparison with the previous one $(x_{\text{mid}-2}, x_{\text{mid}-1})$.

This partition is extended to a grid $\{x_0, x_1, \dots, x_{\text{mid}}, \dots, x_M\}$ of $[0, 1]$ with $M = 2\,\text{mid}$, by setting $x_i = 1 - x_{M-i}$ for $i = \text{mid}+1, \dots, M$. The resulting mesh will be referred as an ε-graded mesh.

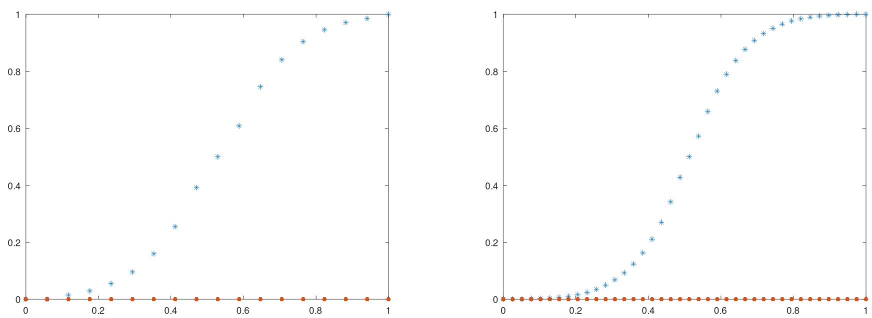

Figure 1. Mesh functions for $\varepsilon = 0.1$, with $M = 17$ and $h = 0.4$ (left) and $M = 39$ and $h = 0.2$ (right).

In order to show the behaviour of the graded meshes near the layers, in figure (3) we plot the nodes x_i against i/M for $i = 0, \ldots, M$.

From now on, we use the letter C to denote a generic constant which is independent of ε, μ and h. Given two quantities A and B the notations $A \lesssim B$ and $A \gtrsim B$ mean that $A \le CB$ and $A \ge CB$ respectively. We also denote by $A \sim B$ when $A \lesssim B$ and $A \gtrsim B$.

Since $x_i - x_{i-1} = hx_{i-1}^{\gamma}$ for $i = 1, \ldots, \text{mid} - 1$ we see that the maximum length of the intervals is like $h(1/2)^{\gamma} \sim h$. Therefore, since M is the number of intervals of the mesh, we have $M \gtrsim 1/h$.

On the other hand, it is proved in [6, proof of Corollary 4.5] that

$$M \lesssim \log\left(\frac{1}{\varepsilon}\right)\frac{1}{h}\log\left(\frac{1}{h}\right),$$

and then using that $M \gtrsim 1/h$, we also obtain

$$h \lesssim \log\left(\frac{1}{\varepsilon}\right)\frac{1}{M}\log M.$$

Hence, we see that h is bounded almost uniformly with respect to ε (up to a logarithmic factor) by the number of elements, similar to the case of quasi-uniform meshes also except for the logarithmic factor $\log M$.

In what follows, we write the error estimates in terms of h, but they can be traduced in terms of the number of degrees of freedom since

$$\frac{1}{M} \lesssim h \lesssim \log\left(\frac{1}{\varepsilon}\right)\frac{1}{M}\log M. \tag{6}$$

Remark 3.1. *We observe that our mesh is constructed first in the interval* $[0, \frac{1}{2}]$ *and then we reflect it into the interval* $\left[\frac{1}{2}, 1\right]$*. Since we enforce the point* $x = \frac{1}{2}$ *to be a node, may be necessary to modify the* *mesh on $\left[0, \frac{1}{2}\right]$ in order to satisfy condition (13), and we can achieve that just by eliminating the node $x_{\mathrm{mid}-1}$ if*

$$x_{\mathrm{mid}} - x_{\mathrm{mid}-1} < \frac{1}{2}\left(x_{\mathrm{mid}-1} - x_{\mathrm{mid}-2}\right).$$

In fact, assuming this, we note that from the definition of the graded mesh we have

$$x_i - x_{i-1} < 2\left(x_{i-1} - x_{i-2}\right), \qquad i = 1, \ldots, \mathrm{mid}-1.$$

Then the last interval after elimination of $x_{\mathrm{mid}-1}$ has length $x_{\mathrm{mid}} - x_{\mathrm{mid}-2}$ and we have

$$x_{\mathrm{mid}} - x_{\mathrm{mid}-2} \leq \frac{3}{2}\left(x_{\mathrm{mid}-1} - x_{\mathrm{mid}-2}\right) < 3\left(x_{\mathrm{mid}-2} - x_{\mathrm{mid}-3}\right).$$

Also

$$x_{\mathrm{mid}} - x_{\mathrm{mid}-2} > x_{\mathrm{mid}-1} - x_{\mathrm{mid}-2} > x_{\mathrm{mid}-2} - x_{\mathrm{mid}-3}.$$

Therefore

$$1 < \frac{x_{\mathrm{mid}} - x_{\mathrm{mid}-2}}{x_{\mathrm{mid}-2} - x_{\mathrm{mid}-3}} < 3$$

as we desired.

3.2. **Lagrange interpolation.** In this Subsection we obtain robust error estimates in the balanced norm for the Lagrange interpolant on ε-graded meshes.

The following two results deal with the interpolation error on the interval $(0,1)$ for functions with the same kind of behavior as stated in Lemma 2.1.

From now on, we assume that $\varepsilon < e^{-2}$, as otherwise the subsequent analysis can be carried out using standard techniques.

Lemma 3.1. *Let $u \in H_0^1(0,1)$ be such that*

$$|u(x)| \leq C_0,$$
$$|u'(x)| \leq C_0\left(1 + \varepsilon^{-1}e^{-\alpha\frac{x}{\varepsilon}} + \varepsilon^{-1}e^{-\alpha\frac{1-x}{\varepsilon}} + \mu^{-1}e^{-\alpha\frac{x}{\mu}} + \mu^{-1}e^{-\alpha\frac{1-x}{\mu}}\right),$$

for all $x \in (0,1)$, with $C_0 \geq 0$ independent of ε and μ. Then, if u^I denotes the piecewise linear Lagrange interpolant of u on an ε-graded mesh, there exists a constant C such that

$$\|u - u^I\|_{0,I} \leq Ch.$$

Proof. Since u is bounded, we have

$$\|u - u^I\|_{0,I_1}^2 \leq 4|I_1|\|u\|_{L^\infty(I_1)}^2 \leq 2C_0|I_1|.$$

Using that $|I_1| = h^s$, we get

$$\|u - u^I\|_{0,I_1}^2 \le Ch^s = Ch^{\frac{1}{1-\gamma}} = Ch^{2\log\frac{1}{\varepsilon}} \le Ch^2,$$

because we have assumed $\varepsilon \le e^{-2}$.

Let

$$b_\varepsilon(x) = 1 + \varepsilon^{-1}e^{-\alpha\frac{x}{\varepsilon}} + \varepsilon^{-1}e^{-\alpha\frac{1-x}{\varepsilon}},$$
$$b_\mu(x) = 1 + \mu^{-1}e^{-\alpha\frac{x}{\mu}} + \mu^{-1}e^{-\alpha\frac{1-x}{\mu}}.$$

Using that for each interval I_i we have

$$\|u - u^I\|_{0,I_i} \le C|I_i|\|u'\|_{0,I_i}$$

and since for $2 \le i \le \text{mid}$ it holds $|I_i| \le hx^\gamma$ for all $x \in I_i$, we obtain

$$\begin{aligned}\|u - u^I\|_{0,\left(0,\frac12\right)\setminus I_1}^2 &\le h^2\|x^\gamma u'\|_{0,\left(0,\frac12\right)}^2 \\ &\le Ch^2\left(h^2 + \|x^\gamma b_\varepsilon\|_{0,\left(0,\frac12\right)}^2 + \|x^\gamma b_\mu\|_{0,\left(0,\frac12\right)}^2\right).\end{aligned}$$

Since $\varepsilon \le \mu$, we have that

$$\gamma = 1 - \frac{1}{2\log\frac{1}{\varepsilon}} \ge 1 - \frac{1}{2\log\frac{1}{\mu}} := \gamma_\mu$$

and therefore $x^\gamma \le x^{\gamma_\mu}$ on $\left(0,\frac12\right)$. Then

$$\|u - u^I\|_{0,\left(0,\frac12\right)\setminus I_1}^2 \le Ch^2\left(1 + \|x^\gamma b_\varepsilon\|_{0,\left(0,\frac12\right)}^2 + \|x^{\gamma_\mu} b_\mu\|_{0,\left(0,\frac12\right)}^2\right).$$

It can be checked that

$$\|x^\gamma b_\varepsilon\|_{0,\left(0,\frac12\right)} \le C \qquad \text{and} \qquad \|x^{\gamma_\mu} b_\mu\|_{0,\left(0,\frac12\right)} \le C. \tag{7}$$

Indeed, for the first inequality, recalling the definition of b_ε, we have

$$\begin{aligned}\|x^\gamma b_{\mathbf{e}}\|_{0,\left(0,\frac12\right)} &\le \left\|x^\gamma\left(1 + 2\varepsilon^{-1}e^{-\alpha\frac{x}{\varepsilon}}\right)\right\|_{0,\left(0,\frac12\right)} \\ &\le C + 2\left\|x^\gamma\varepsilon^{-1}e^{-\alpha\frac{x}{\varepsilon}}\right\|_{0,\left(0,\frac12\right)}.\end{aligned}$$

But, using the substitution $y = x/\varepsilon$, we get

$$\left\|x^\gamma\varepsilon^{-1}e^{-\alpha\frac{x}{\varepsilon}}\right\|_{0,\left(0,\frac12\right)}^2 = \int_0^{\frac12} x^{2\gamma}\varepsilon^{-2}e^{-2\alpha\frac{x}{\varepsilon}}\,dx \le \varepsilon^{2\gamma-1}\int_0^\infty y^{2\gamma}e^{-2\alpha y}\,dx.$$

Since $2\gamma - 1 \ge 0$ for $\varepsilon \le e^{-1}$, we have that the last integral is finite (with the constant involved depending only on α), and we obtain, in this case, the first estimate of (7). The case $\varepsilon > e^{-1}$ is clear. Second estimate in (7) follows similarly.

Therefore, we obtain that

$$\|u - u^I\|_{0,\left(0,\frac12\right)} \le Ch.$$

Clearly a similar estimate can be obtained for the interval $\left(\frac{1}{2},1\right)$. □

Remark 3.2. *With a similar proof, for ε small enough, by using the interpolation error estimate*

$$\|u-u^I\|_{0,I_i} \le C|I_i|^2\|u''\|_{0,I_i}$$

on the intervals $I_i \subset (0,\frac{1}{2})\setminus I_1$, it can be proved that the inequality

$$\|u-u^I\|_{0,I} \le Ch^2$$

holds for ε-graded meshes.

Lemma 3.2. *Let $u \in H^2(0,1)\cap H_0^1(0,1)$ and u^I be its piecewise linear Lagrange interpolation on an ε-graded mesh.*

i) *Suppose $|u''| \le C_0\left(1+\varepsilon^{-2}e^{-\alpha\frac{x}{\varepsilon}}+\mu^{-2}e^{-\alpha\frac{x}{\mu}}+\varepsilon^{-2}e^{-\alpha\frac{1-x}{\varepsilon}}+\mu^{-2}e^{-\alpha\frac{1-x}{\mu}}\right)$ for some constant C_0 independent of ε and μ. Then, we have*

$$\|(u-u^I)'\|_0 \le C\varepsilon^{-\frac{1}{2}}h. \tag{8}$$

ii) *If $|u''| \le C_0\left(1+\mu^{-2}e^{-\alpha\frac{x}{\mu}}+\mu^{-2}e^{-\alpha\frac{1-x}{\mu}}\right)$, with C_0 independent of μ, then*

$$\|(u-u^I)'\|_0 \le C\mu^{-\frac{1}{2}}h. \tag{9}$$

Proof. We will prove the results for the restriction to $\left(0,\frac{1}{2}\right)$ with a boundary layer at $x=0$, the corresponding result for a function with a boundary layer at $x=1$ on the interval $\left(\frac{1}{2},1\right)$ can be obtained by using the same arguments, but estimates on μ instead of ε.

For the first interval I_1, for γ given in (5), we can use the following estimate (see, for example, [16, Proposition 1.2.4])

$$\|(u-u^I)'\|_{0,I_1} \le C|I_1|^{1-\gamma}\|x^\gamma u''\|_{0,I_1},$$

and for the rest of the intervals $I_i, i=1,\dots,\mathrm{mid}$, we have

$$\|(u-u^I)'\|_{0,I_i} \le C|I_i|\|u''\|_{0,I_i}.$$

Then, we obtain

$$\begin{aligned}\|(u-u^I)'\|^2_{0,\left(0,\frac{1}{2}\right)} &= \|(u-u^I)'\|^2_{0,I_1}+\sum_{i=2}^{\mathrm{mid}}\|(u-u^I)'\|^2_{0,I_i}\\ &\le C|I_1|^{2-2\gamma}\|x^\gamma u''\|^2_{0,I_1}+C\sum_{i=2}^{\mathrm{mid}}|I_i|^2\|u''\|^2_{0,I_i}.\end{aligned}$$

Since

$$|I_1| = h^s, \qquad |I_i| = hx_{i-1}^\gamma \quad i=2,\dots,\mathrm{mid}$$

we get

$$
\begin{aligned}
\|(u-u^I)'\|^2_{0,\left(0,\frac{1}{2}\right)} &\le Ch^{2s(1-\gamma)}\|x^\gamma u''\|^2_{0,I_1} + C\sum_{i=2}^{\mathrm{mid}} h^2\|x^\gamma u''\|^2_{0,I_i} \\
&\le Ch^2\|x^\gamma u''\|^2_{0,\left(0,\frac{1}{2}\right)},
\end{aligned}
\tag{10}
$$

since $2s(1-\gamma)=2$.

In order to prove i) we observe that, since the function $f(y)=ye^{-y}$ is decreasing for $y>1$ and the integrals $\int_0^\infty y^\delta e^{-2\alpha y}\,dy$ are uniformly bounded for $\delta\in[0,4]$, taking into account that $\varepsilon\le\mu$, we obtain

$$\|x^\gamma u''\|^2_{0,\left(0,\frac{1}{2}\right)} \le C\left(1+\varepsilon^{2\gamma-3}\right).$$

But $2\gamma-3=-1-\frac{1}{\log\frac{1}{\varepsilon}}$ and $\varepsilon^{2\gamma-3}=e\varepsilon^{-1}$, so

$$\|x^\gamma u''\|^2_{0,\left(0,\frac{1}{2}\right)} \le \frac{C}{\varepsilon}. \tag{11}$$

Thus, by combining (11) and (10), we get

$$\|(u-u^I)'\|^2_{0,\left(0,\frac{1}{2}\right)} \le \frac{C}{\varepsilon}h^2.$$

To obtain ii) we observe that, in this case,

$$\|x^\gamma u''\|^2_{0,\left(0,\frac{1}{2}\right)} \le C\left(1+\mu^{2\gamma-3}\right).$$

Therefore, by using this in (10) and the fact that $\mu^{2\gamma-2}<\varepsilon^{2\gamma-2}$, we obtain

$$\|(u-u^I)'\|^2_{0,\left(0,\frac{1}{2}\right)} \le Ch^2\left(1+\mu^{2\gamma-3}\right) \le Ch^2\mu^{-1}\varepsilon^{2\gamma-2}.$$

Since $\varepsilon^{2\gamma-2}=e$, we have

$$\|(u-u^I)'\|^2_{0,\left(0,\frac{1}{2}\right)} \le C\mu^{-1}h^2,$$

and the proof concludes. □

3.3. **H^1-Stability of L^2 projections.** In this Subsection we define two different L^2 projections and analyze their stability. These results are a fundamental tool in order to obtain our error estimates.

First, for any $u\in H^1(I)$, we consider the typical L^2 projection $\mathcal{P}_0(u)\in V_h$ as

$$\int_I \mathcal{P}_0(u)v = \int_I uv, \quad \forall v\in V_h. \tag{12}$$

The following Lemma provides the stability of $\mathcal{P}_0$ as a map from $H^1(I)$ to V_h.

Lemma 3.3. *Assume that* $\mathcal{T}_h$ *satisfies*

$$\begin{aligned} &|I_{i-1}| \le |I_i| \le c_0 |I_{i-1}|, && 2 \le i \le \text{mid}, \\ &|I_{i+1}| \le |I_i| \le c_0 |I_{i+1}|, && \text{mid} \le i \le M-1, \end{aligned} \tag{13}$$

with $1 \le c_0 < 3$*, then*

$$\|(\mathcal{P}_0 u)'\|_0 \le C \|u'\|_0, \qquad \forall u \in H^1(I),$$

with $C = C(c_0)$.

Proof. Following [4, Theorem 4.1], it is enough to check that condition [4, ineq. (4.2)] is verified.

We define the 2×2 matrices G_ℓ, D_ℓ and H_ℓ by

$$G_\ell[i,j] = (\phi_i^\ell, \phi_j^\ell)_{I_\ell}, \qquad D_\ell = \operatorname{diag}(\|\phi_i^\ell\|^2_{0,I_\ell}), \qquad H_\ell = \operatorname{diag}(\hat{h}_i^\ell).$$

Now, we have to prove that there exists a positive constant c such that

$$\langle H_\ell^{-1} G_\ell H_\ell x^\ell, x^\ell \rangle \ge c \langle D_\ell x^\ell, x^\ell \rangle \quad \forall x^\ell \in \mathbb{R}^2. \tag{14}$$

First, we observe that

$$\left(\phi_i^\ell, \phi_j^\ell\right)_{I_\ell} = \begin{cases} \frac{1}{3} h_\ell & \text{if } i = j \\ \frac{1}{6} h_\ell & \text{if } i = 1, j = 2, \text{ or } i = 2, j = 1. \end{cases} \tag{15}$$

Then it follows that

$$\langle H_\ell^{-1} G_\ell H_\ell x^\ell, x^\ell \rangle = \frac{1}{3} h_\ell (x_1^\ell)^2 + \frac{1}{6} h_\ell \left(\frac{\hat{h}_2^\ell}{\hat{h}_1^\ell} + \frac{\hat{h}_1^\ell}{\hat{h}_2^\ell} \right) x_1^\ell x_2^\ell + \frac{1}{3} h_\ell (x_2^\ell)^2.$$

Now we have,

i) if $\ell = 1$,

$$\begin{aligned} \hat{h}_1^\ell &= h_1, \\ \hat{h}_2^\ell &= \frac{1}{2}(h_1 + h_2) \le \frac{1+c_0}{2} h_1, \\ \hat{h}_2^\ell &\ge h_1, \end{aligned}$$

ii) if $2 \le \ell \le \text{mid}$,

$$\begin{aligned} \hat{h}_1^\ell &= \frac{1}{2}(h_{\ell-1} + h_\ell) \le h_\ell, \\ \hat{h}_2^\ell &= \frac{1}{2}(h_{\ell+1} + h_\ell) \le \frac{1+c_0}{2} h_\ell, \\ \hat{h}_1^\ell &\ge \frac{1+c_0}{2c_0} h_\ell, \\ \hat{h}_2^\ell &\ge h_\ell, \end{aligned}$$

iii) if $\text{mid} + 1 \le \ell \le M - 1$,

$$\begin{aligned}
\hat{h}_1^\ell &= \frac{1}{2}(h_{\ell-1}+h_\ell) \le \frac{1+c_0}{2}h_\ell,\\
\hat{h}_2^\ell &= \frac{1}{2}(h_{\ell+1}+h_\ell) \le h_\ell,\\
\hat{h}_1^\ell &\ge h_\ell,\\
\hat{h}_2^\ell &\ge \frac{1+c_0}{2c_0}h_\ell,
\end{aligned}$$

iv) if $\ell = M$,

$$\begin{aligned}
\hat{h}_1^\ell &= \frac{1}{2}(h_M+h_{M-1}) \le \frac{1+c_0}{2}h_M,\\
\hat{h}_2^\ell &= h_M,\\
\hat{h}_1^\ell &\ge h_M.
\end{aligned}$$

Now, since $c_0 \ge 1$, we have $\frac{1+c_0}{2} + 1 \le 1 + c_0$ and so we get for any ℓ that

$$\frac{\hat{h}_2^\ell}{\hat{h}_1^\ell} + \frac{\hat{h}_1^\ell}{\hat{h}_2^\ell} \le 1 + c_0,$$

and

$$\left| \frac{1}{6}h_\ell \left(\frac{\hat{h}_2^\ell}{\hat{h}_1^\ell} + \frac{\hat{h}_1^\ell}{\hat{h}_2^\ell} \right) x_1^\ell x_2^\ell \right| \le \frac{1}{12}h_\ell(1+c_0)\left[(x_1^\ell)^2 + (x_2^\ell)^2\right].$$

Then, we obtain

$$\langle H_\ell^{-1} G_\ell H_\ell x^\ell, x^\ell \rangle \ge \frac{h_\ell}{12}(3-c_0)[(x_1^\ell)^2 + (x_2^\ell)^2]. \tag{16}$$

On the other hand,

$$\langle D_\ell x^\ell, x^\ell \rangle = \|\phi_1^\ell\|_{0,I_\ell}^2 (x_1^\ell)^2 + \|\phi_2^\ell\|_{0,I_\ell}^2 (x_2^\ell)^2 = \frac{1}{3}h_\ell\left[(x_1^\ell)^2 + (x_2^\ell)^2\right]. \tag{17}$$

Then (14) follows from (16) and (17). □

Now, for any $\mathbf{v}(x) = (v_1(x), v_2(x))^T \in \mathbf{H}_0^1(I)$ we define the projection $Q_h(\mathbf{v}) = (Q_{h,1}(\mathbf{v}), Q_{h,2}(\mathbf{v}))^T \in \mathbf{V}_h$ as

$$(AQ_h\mathbf{v}, \mathbf{v}_h) = (A\mathbf{v}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{18}$$

The projection Q_h is well defined thanks to the positive definiteness (3) of A.

To prove the H^1-stability of the projection Q_h we follow again the ideas of [4]. We need some preliminary definitions and results.

For any interval I_ℓ, we have the following four local vector basis functions of $\mathbf{V}_h$:

$$\mathbf{\Phi}_1^\ell = (\phi_1^\ell, 0)^T, \mathbf{\Phi}_2^\ell = (\phi_2^\ell, 0)^T, \mathbf{\Phi}_3^\ell = (0, \phi_1^\ell)^T, \mathbf{\Phi}_4^\ell = (0, \phi_2^\ell)^T.$$

Now, we introduce the local matrices $\mathbf{G}_\ell, \mathbf{H}_\ell, \mathbf{D}_\ell \in \mathbb{R}^{4\times 4}$ by

$$\begin{aligned} \mathbf{G}_\ell[i,j] &= \left(A\mathbf{\Phi}_i^\ell, \mathbf{\Phi}_j^\ell\right)_{I_\ell} \\ \mathbf{H}_\ell &= \operatorname{diag}\left(\hat{h}_1^\ell, \hat{h}_2^\ell, \hat{h}_1^\ell, \hat{h}_2^\ell\right) \\ \mathbf{D}_\ell &= \operatorname{diag}\left(\|\mathbf{\Phi}_i^\ell\|_{0,I_\ell}^2\right). \end{aligned}$$

We need the following assumption on the coefficient matrix A in order to obtain the stability result.

Assumption 1. *Assume that $\mathcal{T}_h$ satisfies* (13), *with $1 \le c_0 < 3$, and that there exists a positive constant β_0 such that the entries of the matrix A satisfy*

$$(3 - c_0)\, a_{ii}(x) - (2 + c_0)\left(|a_{21}(x)| + |a_{12}(x)|\right) \ge \beta_0, \qquad i = 1, 2$$

for all $x \in [0,1]$.

Lemma 3.4. *Under Assumption 1, there exists a positive constant d_0 such that for h small enough we have*

$$\langle \boldsymbol{H}_\ell^{-1}\boldsymbol{G}_\ell\boldsymbol{H}_\ell\mathbf{x}^\ell, \mathbf{x}^\ell\rangle \ge d_0\langle \boldsymbol{D}_\ell\mathbf{x}^\ell, \mathbf{x}^\ell\rangle \qquad \forall \mathbf{x}^\ell \in \mathbb{R}^4. \tag{19}$$

Proof. A simple computation using the generalized integral mean value theorem and (15) shows that

$$\mathbf{H}_\ell^{-1}\mathbf{G}_\ell\mathbf{H}_\ell = \frac{h_\ell}{6}\begin{pmatrix} 2a_{11}(x_{11}) & \frac{\hat{h}_2}{\hat{h}_1}a_{11}(x'_{11}) & 2a_{21}(x_{21}) & \frac{\hat{h}_2}{\hat{h}_1}a_{21}(x'_{21}) \\ \frac{\hat{h}_1}{\hat{h}_2}a_{11}(x'_{11}) & 2a_{11}(x''_{11}) & \frac{\hat{h}_1}{\hat{h}_2}a_{21}(x'_{21}) & 2a_{21}(x''_{21}) \\ 2a_{12}(x_{12}) & \frac{\hat{h}_2}{\hat{h}_1}a_{12}(x'_{12}) & 2a_{22}(x_{22}) & \frac{\hat{h}_2}{\hat{h}_1}a_{22}(x'_{22}) \\ \frac{\hat{h}_1}{\hat{h}_2}a_{12}(x'_{12}) & 2a_{12}(x''_{12}) & \frac{\hat{h}_1}{\hat{h}_2}a_{22}(x'_{22}) & 2a_{22}(x''_{22}) \end{pmatrix}$$

with x_{ij}, x'_{ij} and x''_{ij}, $i,j = 1,2$, points in I_ℓ, and

$$\mathbf{D}_\ell = \frac{h_\ell}{3}\operatorname{diag}(1,1,1,1).$$

Then, using that for any $a, b \in \mathbb{R}$, $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, and the fact that $a_{12}(x), a_{21}(x) \le 0$ and $a_{11}(x), a_{22}(x) > 0$, $\forall x \in [0,1]$, we have

$$\langle \mathbf{H}_\ell^{-1}\mathbf{G}_\ell\mathbf{H}_\ell\mathbf{x}, \mathbf{x}\rangle \ge \frac{h_\ell}{6}\left(L_1x_1^2 + L_2x_2^2 + L_3x_3^2 + L_4x_4^2\right),$$

with

$$L_1 := \left[2a_{11}(x_{11}) - \frac{1}{2}a_{11}(x'_{11})\frac{\hat{h}_2}{\hat{h}_1} - |a_{21}(x_{21})| - \frac{1}{2}|a_{21}(x'_{21})|\frac{\hat{h}_2}{\hat{h}_1}\right.$$

$$\left.-\frac{1}{2}a_{11}(x'_{11})\frac{\hat{h}_1}{\hat{h}_2} - |a_{12}(x_{12})| - \frac{1}{2}|a_{12}(x'_{12})|\frac{\hat{h}_1}{\hat{h}_2}\right],$$

$$L_2 := \left[2a_{11}(x''_{11}) - \frac{1}{2}a_{11}(x'_{11})\frac{\hat{h}_2}{\hat{h}_1} - |a_{21}(x''_{21})| - \frac{1}{2}|a_{21}(x'_{21})|\frac{\hat{h}_1}{\hat{h}_2}\right.$$

$$\left.- \frac{1}{2}a_{11}(x'_{11})\frac{\hat{h}_1}{\hat{h}_2} - |a_{12}(x''_{12})| - \frac{1}{2}|a_{12}(x'_{12})|\frac{\hat{h}_2}{\hat{h}_1}\right],$$

$$L_3 := \left[2a_{22}(x_{22}) - \frac{1}{2}a_{21}(x'_{21})\frac{\hat{h}_1}{\hat{h}_2} - |a_{21}(x_{21})| - \frac{1}{2}|a_{22}(x'_{22})|\frac{\hat{h}_1}{\hat{h}_2}\right.$$

$$\left.- \frac{1}{2}a_{12}(x'_{12})\frac{\hat{h}_2}{\hat{h}_1} - |a_{12}(x_{12})| - \frac{1}{2}|a_{22}(x'_{22})|\frac{\hat{h}_2}{\hat{h}_1}\right]$$

and

$$L_4 := \left[2a_{22}(x''_{22}) - \frac{1}{2}a_{22}(x'_{22})\frac{\hat{h}_2}{\hat{h}_1} - |a_{21}(x''_{21})| - \frac{1}{2}|a_{21}(x'_{21})|\frac{\hat{h}_2}{\hat{h}_1}\right.$$

$$\left.- \frac{1}{2}a_{12}(x'_{12})\frac{\hat{h}_1}{\hat{h}_2} - |a_{12}(x''_{12})| - \frac{1}{2}|a_{22}(x'_{22})|\frac{\hat{h}_1}{\hat{h}_2}\right].$$

Let $\bar{L}_i(x), i = 1, \ldots, 4$ be defined as L_i but replacing x_{ij}, x'_{ij} and x''_{ij} by x for all i, j. Then our hypothesis implies that for all $x \in [0, 1]$

$$\bar{L}_1(x), \bar{L}_2(x) \geq a_{11}(x)\left(\frac{3}{2} - \frac{1}{2}c_0\right) - \left(1 + \frac{1}{2}c_0\right)(|a_{12}(x)| + |a_{21}(x)|) \geq \frac{\beta_0}{2}$$

and

$$\bar{L}_3(x), \bar{L}_4(x) \geq a_{22}(x)\left(\frac{3}{2} - \frac{1}{2}c_0\right) - \left(1 + \frac{1}{2}c_0\right)(|a_{12}(x)| + |a_{12}(x)|) \geq \frac{\beta_0}{2}.$$

From the uniform continuity of a_{ij} on $[0, 1]$ and the fact that $x_{ij}, x'_{ij}, x''_{ij} \in I_\ell$, with $|I_\ell| \leq h$, it follows that there exists h_0 such that for $h \leq h_0$ we have $L_i \geq \frac{\beta_0}{4}, i = 1 \ldots, 4$ and then

$$\langle \mathbf{H}_\ell^{-1}\mathbf{G}_\ell\mathbf{H}_\ell\mathbf{x}, \mathbf{x}\rangle \geq \frac{\beta_0}{24}h_\ell|\mathbf{x}^\ell|^2.$$

Since

$$\langle \mathbf{D}_\ell\mathbf{x}^\ell, \mathbf{x}^\ell\rangle = \frac{h_\ell}{3}|\mathbf{x}^\ell|^2$$

we have that (19) holds for $h \le h_0$ with $d_0 = \frac{\beta_0}{8}$. □

Now, we define the basis vector functions

$$\mathbf{\Phi}_{2k-1} = (\phi_k, 0)^T, \qquad \mathbf{\Phi}_{2k} = (0, \phi_k)^T, \qquad k = 1, 2, \ldots, M-1.$$

Let $D_\delta \in \mathbb{R}^{(2M-2)\times(2M-2)}$ and $D_\phi \in \mathbb{R}^{(2M-2)\times(2M-2)}$ be the diagonal matrices

$$D_\delta = \operatorname{diag}(\delta_k), \quad D_\phi = \operatorname{diag}\left(\hat{h}_k \|\mathbf{\Phi}_k\|_0\right),$$

with

$$\delta_{2k-1} = \delta_{2k} = \sqrt{\sum_{\ell \in I(k)} h_\ell^{-2} \|\mathbf{\Phi}_k\|_{0,I_\ell}^2}.$$

Finally, we define the $(2M-2)\times(2M-2)$ Gram matrix $\mathbf{G}$ by

$$\mathbf{G}[i,j] = (A\mathbf{\Phi}_i, \mathbf{\Phi}_j)_I .$$

The proof of the next Lemma follows by the same arguments used in [4, Lemma 5.1].

Lemma 3.5. *Under Assumption 1 there exists a positive constant C such that*

$$|\mathbf{x}| \le C|\mathcal{G}\mathbf{x}| \quad \forall \mathbf{x} \in \mathbb{R}^{2M}$$

where $\mathcal{G}$ is the scaled Gram matrix defined by

$$\mathcal{G} = D_\phi^{-1} \mathbf{G} D_\delta^{-1}.$$

Hence, we obtain the following result.

Lemma 3.6. *Under Assumption 1, we have*

$$\sum_{\ell=1}^{M} h_\ell^{-2} (A\mathbf{v}_h, \mathbf{v}_h)_{I_\ell} \lesssim \sum_{k=1}^{2M-2} \left[\frac{(A\mathbf{v}_h, \mathbf{\Phi}_k)}{\hat{h}_k \|\mathbf{\Phi}_k\|_0} \right]^2, \tag{20}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$.

Proof. The proof follows the same ideas given in [4, Lemma 4.1], we include it for the sake of completeness.

Let $\mathbf{v}_h = \sum_{k=1}^{2M-2} \mathbf{v}_h^k \mathbf{\Phi}_k \in \mathbf{V}_h$. For the left hand side of (20) we have

$$\begin{aligned}
\sum_{\ell=1}^{M} h_\ell^{-2} \left(A\mathbf{v}_h, \mathbf{v}_h\right)_{I_\ell} &\leq C \sum_{\ell=1}^{M} h_\ell^{-2} \|\mathbf{v}_h\|_{0,I_\ell}^2 \\
&\leq C \sum_{\ell=1}^{M} h_\ell^{-2} \sum_{k\in J(\ell)} (\mathbf{v}_h^k)^2 \|\mathbf{\Phi}_k\|_{0,\mathcal{T}_\ell}^2 \\
&= C \sum_{k=1}^{2M-2} (\mathbf{v}_h^k)^2 \sum_{\ell\in I(k)} h_\ell^{-2} \|\mathbf{\Phi}_k\|_{0,I_\ell}^2 \\
&= C \sum_{k=1}^{2M-2} (\mathbf{v}_h^k)^2 \delta_k^2 = C \sum_{k=1}^{2M-2} \mathbf{x}_k^2 = C|\mathbf{x}|^2,
\end{aligned}$$

where $\mathbf{x} = (\mathbf{x}_k)_{k=1}^{2M-2} = (\mathbf{v}_h^k \delta_k)_{k=1}^{2M-2}$.

The right hand side in (20) is

$$\begin{aligned}
\sum_{k=1}^{2M-2} \left[\frac{(A\mathbf{v}_h, \mathbf{\Phi}_k)}{\hat{h}_k \|\mathbf{\Phi}_k\|_0}\right]^2 &= \sum_{k=1}^{2M-2} \left[\sum_{j=1}^{2M-2} \mathbf{v}_h^j \frac{(A\mathbf{\Phi}_j, \mathbf{\Phi}_k)}{\hat{h}_k \|\mathbf{\Phi}_k\|_0}\right]^2 \\
&= \sum_{k=1}^{2M-2} \left[\sum_{j=1}^{2M-2} \underline{x}_j \frac{(A\mathbf{\Phi}_j, \mathbf{\Phi}_k)}{\delta_j \hat{h}_k \|\mathbf{\Phi}_k\|_0}\right]^2 \\
&= \sum_{k=1}^{2M-2} \left[(\mathcal{G}\mathbf{x})_k\right]^2 = |\mathcal{G}\mathbf{x}|^2.
\end{aligned}$$

Therefore, (20) follows from the previous Lemma. □

Now, we are able to prove the H^1-stability of the Q_h-projection. The proof follows the lines of [4, Theorem 4.1].

Theorem 3.1. *If Assumption 1 holds, then the operator Q_h is $\mathbf{H}^1(I)$-stable, i.e.,*

$$\|Q_h\mathbf{v}\|_1 \leq C\|\mathbf{v}\|_1 \quad \textit{for all } \mathbf{v} \in \mathbf{H}^1(I),$$

with C a positive constant independent of $\mathbf{v}$ and h.

Proof. We define $\Pi_h\mathbf{v} = (v_1^I, v_2^I)^T$ where v_j^I denotes the classical Lagrange interpolant of v_j in V_h, $j = 1, 2$.

From the triangle inequality, the H^1-stability of the classical Lagrange interpolant in V_h and the classical local inverse inequality $\|v_h\|_{1,I_\ell} \leq h_\ell^{-1}\|v_h\|_{0,I_\ell}$ for any $v_h \in V_h$, it follows that there exists a constant C

such that

$$\|Q_h\mathbf{v}\|_1^2 \le C\left(\|\Pi_h\mathbf{v}\|_1^2 + \sum_{\ell=1}^{M}\|Q_h\mathbf{v}-\Pi_h\mathbf{v}\|_{1,I_\ell}^2\right) \le C\left(\|\mathbf{v}\|_1^2 + \sum_{\ell=1}^{M} h_\ell^{-2}\|Q_h\mathbf{v}-\Pi_h\mathbf{v}\|_{0,I_\ell}^2\right).$$

Now, using (3) we get that $\|\mathbf{v}\|_{0,I_\ell}^2$ is equivalent to $(A\mathbf{v},\mathbf{v})_{I_\ell}$.

Hence, we obtain

$$\sum_{\ell=1}^{M} h_\ell^{-2}\|Q_h\mathbf{v}-\Pi_h\mathbf{v}\|_{0,I_\ell}^2 \le C\sum_{\ell=1}^{M} h_\ell^{-2}\left(A(Q_h\mathbf{v}-\Pi_h\mathbf{v}), Q_h\mathbf{v}-\Pi_h\mathbf{v}\right)_{I_\ell}.$$

Denote by ω_k the support of ϕ_k, that is, $\omega_k = I_k \cup I_{k+1}$. From the Lemma above and the Schwarz inequality, we can conclude that

$$\begin{aligned}
\sum_{\ell=1}^{M} h_\ell^{-2}\|Q_h\mathbf{v}-\Pi_h\mathbf{v}\|_{0,I_\ell}^2 &\le C\sum_{k=1}^{2M}\left[\frac{(A(Q_h\mathbf{v}-\Pi_h\mathbf{v}),\mathbf{\Phi}_k)}{h_k\|\mathbf{\Phi}_k\|_0}\right]^2 \\
&= C\sum_{k=1}^{2M}\left[\frac{(A(\mathbf{v}-\Pi_h\mathbf{v}),\mathbf{\Phi}_k)_{\omega_k}}{h_k\|\mathbf{\Phi}_k\|_0}\right]^2 \\
&= C\sum_{k=1}^{2M} h_k^{-2}\|A(\mathbf{v}-\Pi_h\mathbf{v})\|_{0,\omega_k}^2 \\
&\le C\sum_{k=1}^{2M}\|\mathbf{v}\|_{1,\omega_k}^2 \le C\|\mathbf{v}\|_1^2,
\end{aligned}$$

where we use again the Lagrange interpolation error estimates. □

3.4. **An auxiliary estimate for a single reaction-diffusion equation.** In this subsection we present, as a preliminary result, error estimates in balanced norms for a singularly perturbed reaction-diffusion equation.

Let us consider the reaction-diffusion equation

$$\begin{aligned} -\mu^2 u''(x) + b_0 u(x) &= f(x) \quad x\in(0,1) \\ u(0) = u(1) &= 0 \end{aligned} \tag{21}$$

with $\mu \ge \varepsilon$, b_0 a positive constant and f smooth. It is well known that the solution satisfies (see [9])

$$|u^{(k)}(x)| \le C\left(1+\mu^{-k}e^{-b_0\frac{x}{\mu}}+\mu^{-k}e^{-b_0\frac{1-x}{\mu}}\right), \quad k=0,1,2. \tag{22}$$

The energy norm associated to the problem (21) is given by

$$\|u\|_e^2 = \mu^2\|u'\|_0^2 + b_0\|u\|_0^2,$$

and the corresponding balanced norm is given by

$$|||u|||_b^2 = \mu\|u'\|_0^2 + b_0\|u\|_0^2.$$

From the definition of $\mathcal{P}_0$ given in equation (12) and since $\mu^2(u' - u_h', v') + b_0(u - u_h, v) = 0, \forall v \in V_h$, following [13, Subsection 2.3.1] we get

$$\begin{aligned}\|u_h - \mathcal{P}_0 u\|_e^2 &= \mu^2 \int_0^1 (u - \mathcal{P}_0 u)'(u_h - \mathcal{P}_0 u)' \\ &\le \mu^2\|(u - \mathcal{P}_0 u)'\|_0\|(u_h - \mathcal{P}_0 u)'\|_0 \\ &\le \mu\|(u - \mathcal{P}_0 u)'\|_0\|u_h - \mathcal{P}_0 u\|_e.\end{aligned}$$

Thus,

$$\mu\|(u_h - \mathcal{P}_0 u)'\|_0 \le \|u_h - \mathcal{P}_0 u\|_e \le \mu\|(u - \mathcal{P}_0 u)'\|_0,$$

and, in particular,

$$\|(u_h - \mathcal{P}_0 u)'\|_0 \le \|(u - \mathcal{P}_0 u)'\|_0. \tag{23}$$

By the triangular inequality and (23)

$$\begin{aligned}\mu^{\frac{1}{2}}\|(u - u_h)'\|_0 &\le \mu^{\frac{1}{2}}\|(u - \mathcal{P}_0 u)'\|_0 + \mu^{\frac{1}{2}}\|(\mathcal{P}_0 u - u_h)'\|_0 \\ &\le 2\mu^{\frac{1}{2}}\|(u - \mathcal{P}_0 u)'\|_0.\end{aligned}$$

Therefore, if we prove that

$$\|(u - \mathcal{P}_0 u)'\|_0 \le C\mu^{-\frac{1}{2}}h$$

we would obtain

$$\mu^{\frac{1}{2}}\|(u - u_h)'\|_0 \le Ch.$$

Let $u^I \in V_h$ be the Lagrange interpolant of the solution u of (21). Then, from H^1-stability of the projection $\mathcal{P}_0$ (Lemma 3.3), we get

$$\begin{aligned}\|(u - \mathcal{P}_0 u)'\|_0 &\le \|(u - u^I)'\|_0 + \|[\mathcal{P}_0(u - u^I)]'\|_0 \\ &\le (1 + C)\|(u - u^I)'\|_0.\end{aligned}$$

Then, in view of (22) and the error estimate for Lagrange interpolation given in Lemma 3.2, we have that $\|(u - u^I)'\|_0 \le C\mu^{-\frac{1}{2}}h$. So

$$\|(u - \mathcal{P}_0 u)'\|_0 \le C\mu^{-\frac{1}{2}}h,$$

and we have the following result.

Theorem 3.2. *Let u be the solution of the problem* (21)*. Let $u_h \in V_h$ the solution of $-\mu^2(u_h', v') + b_0(u_h, v) = (f, v), \forall v \in V_h$. Assuming that the ε-graded mesh satisfies* (13) *we have that*

$$\|(u-u_h)'\|_0 \le C\mu^{-\frac{1}{2}}h.$$

As a consequence of the last theorem and the Lagrange interpolation error estimates for the solution u, we have

$$\|u-u_h\|_0 \le \|u-u_h\|_e \le Ch,$$

and therefore, we obtain the following estimate for the error in the balanced norm

$$|||u-u_h|||_b^2 = \mu\|(u-u_h)'\|_0^2 + b_0\|u-u_h\|_0^2 \le Ch^2.$$

4. Error estimates in balanced norms

The goal of this Section is to obtain error estimates for the solution of the coupled system (1) by using the ε-graded meshes introduced in the previous Section.

First, we analyze a coupled system with equal perturbed parameters. Then, we consider the case of a coupled system with two different small parameters but assuming constant coefficients in both equations.

4.1. **Case $\mu = \varepsilon$.**

Theorem 4.1. *Let $\mathbf{u} = (u_1, u_2)$ be the solution of the system* (1) *with $\varepsilon = \mu$, and let $\mathbf{u}_h = (u_{h,1}, u_{h,2})$ be its corresponding finite element approximation on $\mathbf{V}_h$. Assuming Assumption 1 holds and that ε-graded mesh satisfies the condition* (13)*, we have that there exists a constant C, independent of ε, such that*

$$|||\mathbf{u}-\mathbf{u}_h||| \le Ch.$$

Proof. Following the same ideas of the proof of Theorem 3.2, we observe that, if we consider the Q_h projector defined in (18) we have

$$\begin{aligned} C\|\mathbf{u}_h - Q_h\mathbf{u}\|_e^2 &\le \mathbf{B}(\mathbf{u}_h - Q_h\mathbf{u}, \mathbf{u}_h - Q_h\mathbf{u}) = \mathbf{B}(\mathbf{u} - Q_h\mathbf{u}, \mathbf{u}_h - Q_h\mathbf{u}) \\ &= \varepsilon^2 \int_0^1 (\mathbf{u} - Q_h\mathbf{u})' \cdot (\mathbf{u}_h - Q_h\mathbf{u})' \\ &\le \varepsilon\|(\mathbf{u} - Q_h\mathbf{u})'\|_0 \|\mathbf{u}_h - Q_h\mathbf{u}\|_e. \end{aligned}$$

Therefore, $C\|\mathbf{u}_h - Q_h\mathbf{u}\|_e \le \varepsilon\|(\mathbf{u} - Q_h\mathbf{u})'\|_0$. Moreover, since $\varepsilon\|(\mathbf{u}_h - Q_h\mathbf{u})'\|_0 \le \|\mathbf{u}_h - Q_h\mathbf{u}\|_e$, we can conclude that

$$\|(\mathbf{u}_h - Q_h\mathbf{u})'\|_0 \le C\|(\mathbf{u} - Q_h\mathbf{u})'\|_0.$$

Then, using the Lagrange interpolant $\Pi_h \mathbf{u} = (u_1^I, u_2^I)^T$ and the H^1-stability of Q_h obtained in Theorem 3.1 together with Poincare's inequality, we get

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}\|(\mathbf{u}-\mathbf{u}_h)'\|_0 &\le \varepsilon^{\frac{1}{2}}\|\,(\mathbf{u}-Q_h\mathbf{u})'\,\|_0 + \varepsilon^{\frac{1}{2}}\|\,(Q_h\mathbf{u}-\mathbf{u}_h)'\,\|_0 \\
&\le C\varepsilon^{\frac{1}{2}}\|\,(\mathbf{u}-Q_h\mathbf{u})'\,\|_0 \\
&\le C\varepsilon^{\frac{1}{2}}\left\{\|\,(\mathbf{u}-\Pi_h\mathbf{u})'\,\|_0 + \|\,[Q_h\,(\mathbf{u}-\Pi_h\mathbf{u})]'\,\|_0\right\} \\
&\le C\varepsilon^{\frac{1}{2}}\|\,(\mathbf{u}-\Pi_h\mathbf{u})'\,\|_0.
\end{aligned}$$

Therefore, from Lemma 3.2, we obtain

$$\varepsilon^{\frac{1}{2}}\|(\mathbf{u}-\mathbf{u}_h)'\|_0 \le Ch. \tag{24}$$

On the other hand, from the Galerkin orthogonality and the Lagrange interpolation error estimates of Lemmas 3.1 and 3.2, we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le \|\mathbf{u}-\mathbf{u}_h\|_e \le \|\mathbf{u}-\Pi_h\mathbf{u}\|_e \le Ch. \tag{25}$$

Thus, the proof concludes from inequalities (24) and (25). □

Remark 4.1. *We want to observe that our results can be extended, by analogous arguments, to the case of more than two equations. For example, for the case of three equations the main results can be obtained by assuming that the entries of the matrix A satisfy that $a_{ii} > 0, a_{ij} \le 0, i \ne j$, $1 \le i, j \le 3$, there exists $\alpha \ne 0$ such that*

$$\min_{x\in[0,1]}\left\{\sum_{j=1}^{3} a_{ij}(x), 1 \le i \le 3\right\} \ge \alpha^2$$

and

$$(3-c_0)\,a_{ii}(x) - (2+c_0)\sum_{j\ne i}(|a_{ji}(x)| + |a_{ij}(x)|) \ge \beta_0, \qquad i = 1,2,3$$

for all $x \in [0,1]$.

4.2. **Case $\varepsilon \ne \mu$.** In this Subsection we analyze the case in which we have two different small perturbed parameters but all the entries of the matrix A are constants.

We observe that, if we consider again the projector $Q_h\mathbf{u} = (Q_{h,1}\mathbf{u}, Q_{h,2}\mathbf{u})^T$ we have that

$$\begin{aligned}
\int_I (a_{11}Q_{h,1}\mathbf{u} + a_{12}Q_{h,2}\mathbf{u})\,v_1 &= \int_I (a_{11}u_1 + a_{12}u_2)\,v_1 \qquad \forall v_1 \in V_h, \\
\int_I (a_{21}Q_{h,1}\mathbf{u} + a_{22}Q_{h,2}\mathbf{u})\,v_2 &= \int_I (a_{21}u_1 + a_{22}u_2)\,v_2 \qquad \forall v_2 \in V_h,
\end{aligned}$$

and so

$$
\begin{aligned}
C\|\mathbf{u}_h - Q_h\mathbf{u}\|_e^2 &\le \mathbf{B}(\mathbf{u}_h - Q_h\mathbf{u}, \mathbf{u}_h - Q_h\mathbf{u}) = \mathbf{B}(\mathbf{u} - Q_h\mathbf{u}, \mathbf{u}_h - Q_h\mathbf{u}) \\
&= \varepsilon^2 \int_0^1 (u_1 - Q_{h,1}\mathbf{u})'(u_{h,1} - Q_{h,1}\mathbf{u})' + \\
&\qquad \mu^2 \int_0^1 (u_2 - Q_{h,2}\mathbf{u})'(u_{h,2} - Q_{h,2}\mathbf{u})' \\
&\le \varepsilon^2 \|(u_1 - Q_{h,1}\mathbf{u})'\|_0 \|(u_{h,1} - Q_{h,1}\mathbf{u})'\|_0 + \\
&\qquad \mu^2 \|(u_2 - Q_{h,2}\mathbf{u})'\|_0 \|(u_{h,2} - Q_{h,2}\mathbf{u})'\|_0 \\
&\le \left[\varepsilon\|(u_1 - Q_{h,1}\mathbf{u})'\|_0 + \mu\|(u_2 - Q_{h,2}\mathbf{u})'\|_0\right] \|\mathbf{u}_h - Q_h\mathbf{u}\|_e.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\varepsilon\|(u_{h,1} - Q_{h,1}\mathbf{u})'\|_0 + \mu\|(u_{h,2} - Q_{h,2}\mathbf{u})'\|_0 \le \\
&\qquad C\|\mathbf{u}_h - Q_h\mathbf{u}\|_e \le \varepsilon\|(u_1 - Q_{h,1}\mathbf{u})'\|_0 + \mu\|(u_2 - Q_{h,2}\mathbf{u})'\|_0,
\end{aligned}
$$

and, in particular, we have

$$
\begin{aligned}
\varepsilon\|(u_{h,1} - Q_{h,1}\mathbf{u})'\|_0 &\le \varepsilon\|(u_1 - Q_{h,1}\mathbf{u})'\|_0 + \mu\|(u_2 - Q_{h,2}\mathbf{u})'\|_0, \\
\mu\|(u_{h,2} - Q_{h,2}\mathbf{u})'\|_0 &\le \varepsilon\|(u_1 - Q_{h,1}\mathbf{u})'\|_0 + \mu\|(u_2 - Q_{h,2}\mathbf{u})'\|_0.
\end{aligned}
$$

We observe that, in view of the previous results, we might expect

$$
\begin{aligned}
\|(u_1 - Q_{h,1}\mathbf{u})'\|_0 &\le C\varepsilon^{-\frac{1}{2}} h, \\
\|(u_2 - Q_{h,2}\mathbf{u})'\|_0 &\le C\mu^{-\frac{1}{2}} h,
\end{aligned}
$$

however, we could only get

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}}\|(u_{h,1} - Q_{h,1}\mathbf{u})'\|_0 &\le Ch\left[1 + \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}\right], \\
\mu^{\frac{1}{2}}\|(u_{h,2} - Q_{h,2}\mathbf{u})'\|_0 &\le Ch\left[1 + \left(\frac{\varepsilon}{\mu}\right)^{\frac{1}{2}}\right],
\end{aligned}
$$

and so these would not give the desired estimate for $\varepsilon^{\frac{1}{2}}\|(u_{h,1} - Q_{h,1}\mathbf{u})'\|_0$.

In order to get the optimal estimations in the balanced norm, we use a trick introduced by Roos [22] defining the projection $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \mathbf{V}_h$ as: $\forall v_1, v_2 \in V_h$,

(26)

$$
\begin{aligned}
(a_{11}\hat{u}_1(x) + a_{12}\hat{u}_2(x), v_1) &= (a_{11}u_1(x) + a_{12}u_2(x), v_1), \\
\mu^2(\hat{u}_2', v_2') + (a_{21}\hat{u}_1(x) + a_{22}\hat{u}_2(x), v_2) &= \mu^2(u_2', v_2') + (a_{21}u_1(x) + a_{22}u_2(x), v_2).
\end{aligned}
$$

From the first equation of (26) we have

$$
(\hat{u}_1(x), v_1) = \left(u_1(x) + \frac{a_{12}}{a_{11}}(u_2(x) - \hat{u}_2(x)), v_1\right), \qquad \forall v_1 \in V_h,
$$

and so we can eliminate $\hat{u}_1$ in the second equation of (26) and get

$$\mu^2(\hat{u}_2', v_2') + \left(a_{21}\left[u_1(x) + \frac{a_{12}}{a_{11}}(u_2(x) - \hat{u}_2(x))\right] + a_{22}\hat{u}_2(x), v_2\right) = \mu^2(u_2', v_2') + (a_{21}u_1(x) + a_{22}u_2(x), v_2), \qquad \forall v_2 \in V_h.$$

Therefore,

$$\mu^2(\hat{u}_2', v_2') + \left[\left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)\hat{u}_2(x), v_2\right] = \mu^2(u_2', v_2') + \left[\left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)u_2(x), v_2\right], \qquad \forall v_2 \in V_h. \tag{27}$$

Let $c_A = a_{22} - a_{21}\frac{a_{12}}{a_{11}}$, which satifies

$$c_A = \frac{a_{22}a_{11} - a_{21}a_{12}}{a_{11}} = \frac{det(A)}{a_{11}} > 0.$$

Then, the equation (27) can be written as

$$\mu^2(\hat{u}_2', v_2') + (c_A\hat{u}_2, v_2) = \mu^2(u_2', v_2') + (c_A u_2, v_2), \qquad \forall v_2 \in V_h.$$

Thus, $\hat{u}_2$ is indeed the projection on V_h of u_2 with the inner product $a_\mu(u, v) = \mu^2(u', v') + c_A(u, v)$, i.e.,

$$a_\mu(u_2 - \hat{u}_2, v) = 0, \qquad \forall v \in V_h.$$

Then, taking into account that

$$(\mathcal{P}_0 u_2, v) = (u_2, v), \qquad \forall v \in V_h,$$

we can proceed as in the proof of Theorem 3.2 and observe that

$$\|\hat{u}_2 - \mathcal{P}_0 u_2\|_e \leq \mu\|(u_2 - \mathcal{P}_0 u_2)'\|_0,$$

from which, since $\mu\|\hat{u}_2 - \mathcal{P}_0 u_2\|_0 \leq \|\hat{u}_2 - \mathcal{P}_0 u_2\|_e$, we get

$$\|(\hat{u}_2 - \mathcal{P}_0 u_2)'\|_e \leq C\|(u_2 - \mathcal{P}_0 u_2)'\|_0.$$

Hence, using the triangle inequality and the stability result for $\mathcal{P}_0$ in Lemma 3.3, we can conclude that

$$\mu^{\frac{1}{2}}\|(u_2 - \hat{u}_2)'\|_0 \leq C\mu^{\frac{1}{2}}\|(u_2 - \mathcal{P}_0 u_2)'\|_0 \leq C\mu^{\frac{1}{2}}\|(u_2 - u_2^I)'\|_0.$$

Then, from (9), we finally obtain

$$\mu^{\frac{1}{2}}\|(u_2 - \hat{u}_2)'\|_0 \leq Ch.$$

Now, we want to obtain an error estimate for $\varepsilon\|(u_1 - \hat{u}_1)'\|_0^2$.

For any $v_h \in V_h$, we have that

$$(u_1, v_h) = \left(\hat{u}_1 + \frac{a_{12}}{a_{11}}(\hat{u}_2 - u_2), v_h\right) = \left(\hat{u}_1 + \frac{a_{12}}{a_{11}}(\hat{u}_2 - \mathcal{P}_0(u_2)), v_h\right).$$

As a consequence of the uniqueness of the L^2 projection $\mathcal{P}_0$ we can affirm that

$$\hat{u}_1 + \frac{a_{12}}{a_{11}}(\hat{u}_2 - \mathcal{P}_0(u_2)) = \mathcal{P}_0(u_1),$$

i.e.,

$$\hat{u}_1 = \mathcal{P}_0(u_1) - \frac{a_{12}}{a_{11}}(\hat{u}_2 - \mathcal{P}_0(u_2)).$$

Thus, from Lemmas 2.1 and 3.2, we can assure that

$$\begin{aligned}\|(u_1 - \hat{u}_1)'\|_0 &= \left\|\left(u_1 - \mathcal{P}_0(u_1) + \frac{a_{12}}{a_{11}}\left(\hat{u}_2 - \mathcal{P}_0(u_2)\right)\right)'\right\|_0 \\ &\le \|(u_1 - \mathcal{P}_0(u_1))'\|_0 + C\|(\hat{u}_2 - u_2)'\|_0 + \|(u_2 - \mathcal{P}_0(u_2))'\|_0 \\ &\le C\varepsilon^{-\frac{1}{2}}h + C\mu^{-\frac{1}{2}}h.\end{aligned}$$

Therefore, $\varepsilon^{\frac{1}{2}}\|(\hat{u}_1 - u_1)'\|_0 \le Ch$. Now, from the definition (26) we can write

$$\begin{aligned}\varepsilon^2\|(\hat{u}_1 - u_{h,1})'\|_0^2 &\le \mathbf{B}(\hat{\mathbf{u}} - \mathbf{u}_h, \hat{\mathbf{u}} - \mathbf{u}_h) = \mathbf{B}(\hat{\mathbf{u}} - \mathbf{u}, \hat{\mathbf{u}} - \mathbf{u}_h) \\ &= \varepsilon^2\int_I (\hat{u}_1 - u_{h,1})'(\hat{u}_1 - u_1)' \le \varepsilon^2\|(\hat{u}_1 - u_{h,1})'\|_0\|(\hat{u}_1 - u_1)'\|_0.\end{aligned}$$

Therefore,

$$\|(\hat{u}_1 - u_{h,1})'\|_0 \le \|(\hat{u}_1 - u_1)'\|_0 \le \varepsilon^{-\frac{1}{2}}Ch,$$

and, as a consequence,

$$\varepsilon^{\frac{1}{2}}\|(u_1 - u_{h,1})'\|_0 \le \varepsilon^{\frac{1}{2}}\left\{\|(\hat{u}_1 - u_1)'\|_0 + \|(\hat{u}_1 - u_{1,h})'\|_0\right\} \le Ch. \tag{28}$$

On the other hand, since

$$\begin{aligned}\mu^2\|(\hat{u}_2 - u_{h,2})'\|_0^2 &\le \mathbf{B}(\hat{\mathbf{u}} - \mathbf{u}_h, \hat{\mathbf{u}} - \mathbf{u}_h) = \mathbf{B}(\hat{\mathbf{u}} - \mathbf{u}, \hat{\mathbf{u}} - \mathbf{u}_h) \\ &= \varepsilon^2\int_I (\hat{u}_1 - u_{h,1})'(\hat{u}_1 - u_1)' \le \varepsilon^2\|(\hat{u}_1 - u_{h,1})'\|_0\|(\hat{u}_1 - u_1)'\|_0,\end{aligned}$$

we get

$$\mu^2\|(\hat{u}_2 - u_{h,2})'\|_0^2 \le \varepsilon^2 C\varepsilon^{-1}h^2 = C\varepsilon h^2,$$

and so,

$$\mu\|(\hat{u}_2 - u_{h,2})'\|_0^2 \le C\frac{\varepsilon}{\mu}h^2 \le Ch^2.$$

Consequently,

$$\mu\|(u_2 - u_{h,2})'\|_0^2 \le \mu\|(u_2 - \hat{u}_2)'\|_0^2 + \mu\|(\hat{u}_2 - u_{h,2})'\|_0^2 \le Ch^2. \tag{29}$$

On the other hand, from Lemma 3.1 and Lemma 3.2, it is clear that

$$\|\mathbf{u} - \mathbf{u}_0\|_0 \le \|\mathbf{u} - \mathbf{u}_h\|_e \le \|\mathbf{u} - \Pi\mathbf{u}\|_e \le Ch. \tag{30}$$

So, as an immediate consequence of estimations (28), (29) and (30), we obtain the following main result.

Theorem 4.2. *Let $\mathbf{u} = (u_1, u_2)$ be the solution of the system (1) and let $\mathbf{u}_h = (u_{h,1}, u_{h,2})$ be its corresponding finite element approximation on $\mathbf{V}_h$. Assuming that the ε-graded mesh satisfies the condition (13), we have that there exists a constant C, independent of ε and μ, such that*

$$|||\mathbf{u} - \mathbf{u}_h||| \leq Ch.$$

Remark 4.2. *We note that the case of more than two equations, with constant coefficients and different perturbation parameters, can be treated with similar arguments. Indeed, in the case of three equations, with perturbation parameters $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_3$ and a positive definite matrix $A \in \mathbb{R}^{3\times 3}$, the projection $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \mathbf{V}_h$ can be defined as in (26) by asking*

$$\begin{aligned}
(a_{11}\hat{u}_1(x) + a_{12}\hat{u}_2(x) \; &+ a_{13}\hat{u}_3(x), v_1) = \\
&(a_{11}u_1(x) + a_{12}u_2(x) + a_{13}u_3(x), v_1) \\
(a_{21}\hat{u}_1(x) + a_{22}\hat{u}_2(x) \; &+ a_{23}\hat{u}_3(x), v_2) = \\
&(a_{21}u_1(x) + a_{22}u_2(x) + a_{23}u_3(x), v_2) \\
\varepsilon_3^2(\hat{u}_3', v_3') + &(a_{31}\hat{u}_1(x) + a_{32}\hat{u}_2(x) + a_{33}\hat{u}_3(x), v_3) = \\
&\varepsilon_3^2(u_3', v_3') + (a_{31}u_1(x) + a_{32}u_2(x) + a_{33}u_3(x), v_3)
\end{aligned}$$

for all $v_1, v_2, v_3 \in V_h$.

5. Numerical examples

In this Section, we present numerical examples that confirm the theoretical results of Theorems 4.1 and 4.2.

Problems are approximated using graded meshes as specified in Subsection 3.1 for the corresponding parameters ε and h. The errors $\mathbf{e}_h = |||\mathbf{u} - \mathbf{u}_h|||$ are computed numerically, comparing the approximated solution $\mathbf{u}_h$, for h from 0.32 to 0.01, with the finite element solution obtained with a mesh ten times finer, i.e. with $h = 10^{-3}$.

We compute the rates of convergence with respect to h, r_h, and with respect to the number of degrees of freedom, r_N, as follows:

$$r_h = \frac{\log \mathbf{e}_h - \log \mathbf{e}_{\frac{h}{2}}}{\log 2}, \qquad r_N = -\frac{\log \mathbf{e}_h - \log \mathbf{e}_{\frac{h}{2}}}{\log N_h - \log N_{\frac{h}{2}}},$$

where N_h denotes the number of elements of an ε-graded mesh $\mathcal{T}_h$.

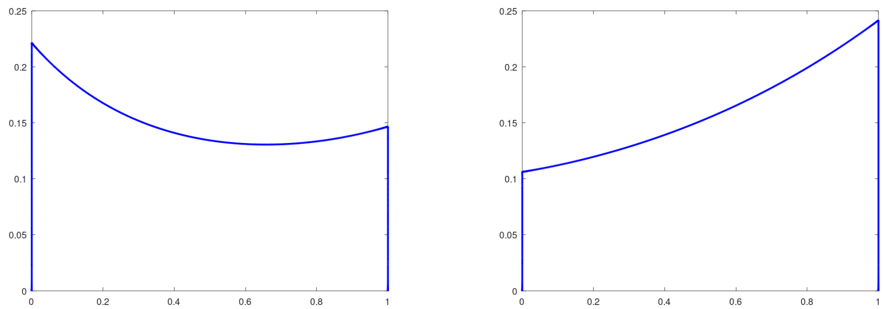

FIGURE 2. Numerical solutions u_1 (left) and u_2 (right) of Example 5.1 for $\varepsilon = 10^{-6}$.

h	$\varepsilon = 10^{-6}, \mu = 10^{-2}$				$\varepsilon = 10^{-9}, \mu = 10^{-3}$			
	N_h	$\vert\vert\vert\cdot\vert\vert\vert$-error	r_h	r_N	N_h	$\vert\vert\vert\cdot\vert\vert\vert$-error	r_h	r_N
0.32	144	3.5080e-02	-	-	216	3.4808e-02	-	-
0.16	330	1.9185e-02	0.87	0.73	498	1.9040e-02	0.87	0.72
0.08	684	1.0118e-02	0.92	0.88	1036	1.0043e-02	0.92	0.87
0.04	1376	5.2127e-03	0.96	0.95	2082	5.1746e-03	0.96	0.95
0.02	2742	2.6544e-03	0.97	0.98	4148	2.6351e-03	0.97	0.98
0.01	5456	1.3527e-03	0.97	0.98	8252	1.3429e-03	0.97	0.98

TABLE 1. Numerical errors for Example 5.1

The computations were implemented in Octave [7]. We remark that the computation of the integrals involving in the bilinear form $\mathbf{B}$ can be made just using the mesh widths. The linear system was solved using the *backslash* "\" command.

5.1. **First Example: variable coefficients, $\varepsilon = \mu$.** In order to confirm the estimates given in Theorem 4.1, in this example we consider the numerical solution of system (1) with variable coefficients given by

$$\begin{aligned} a_{11}(x) &= 5(x+1)^2, & a_{12}(x) &= -(1+x)^3, \\ a_{21}(x) &= -2\cos\left(\frac{\pi}{4}x\right), & a_{22}(x) &= 5e^{1-x}, \end{aligned} \tag{31}$$

and

$$f_1(x) = f_2(x) = 1 \qquad \text{on } [0,1].$$

A graph of the numerical solution obtained for $\varepsilon = 10^{-6}$ is shown in Figure 2. Table 1 reports the errors in balanced norm and the numerical rates of convergence obtained for $\varepsilon = 10^{-6}$ and $\varepsilon = 10^{-9}$. The purpose of Table 2 is to study the robustness of graded meshes

ε	Column A	Column B
	$\vert\vert\vert\cdot\vert\vert\vert$-error	$\vert\vert\vert\cdot\vert\vert\vert$-error
1e-01	1.4193e-02	8.1381e-03
1e-02	1.3104e-02	8.6045e-03
1e-03	1.2767e-02	9.0517e-03
1e-04	1.2616e-02	9.5381e-03
1e-05	1.2528e-02	1.0051e-02
1e-06	1.2471e-02	1.0591e-02
1e-07	1.2431e-02	1.1158e-02
1e-08	1.2401e-02	1.1753e-02
1e-09	1.2378e-02	1.2378e-02

TABLE 2. Comparison of estimated errors in balanced norm, for Example 5.1, for different values of ε. Column A: graded meshes for particular ε and $h = 0.1$ are used in each case. Column B: a single graded mesh for $\varepsilon = 10^{-9}$ and $h = 0.1$ is used for all cases

in two aspects. Firstly, the parameter $h = 0.1$ is fixed while ε varies between 10^{-9} and 10^{-1}. Solution are computed using each ε-graded mesh with $h = 0.1$. We see in Column A that the estimated numerical errors in balanced norm remain in a stable range. Secondly, a single ε-graded mesh, designed for fixed parameters $\varepsilon = 10^{-9}$ and $h = 0.1$, is used to compute the solution for problems with different values of the parameter ε. We see in Column B that the numerical errors also remain almost unchanged near 0.01.

5.2. **Second Example: constant coefficients, $\varepsilon \neq \mu$.** In order to confirm the results of Theorem 4.2 we consider the following coupled reaction–diffusion problem with constant coefficients, taken from [14, 17]:

$$\begin{cases} -\varepsilon^2 u_1''(x) + 2u_1(x) - u_2(x) = 1 & \text{in } I := (0,1) \\ -\mu^2 u_2''(x) - u_1(x) + 2u_2(x) = 1 & \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0. & \end{cases} \tag{32}$$

Figure 5.1 shows the graphs of the numerical solutions u_1 and u_2 which matches with those presented in [17]. One can observe the structure of the boundary layers when different parameters ε and μ are considered.

In Table 3 we show the numerical results for the approximation of problem (32) for the cases $\varepsilon = 10^{-6}, \mu = 10^{-2}$ and $\varepsilon = 10^{-9}, \mu = 10^{-3}$.

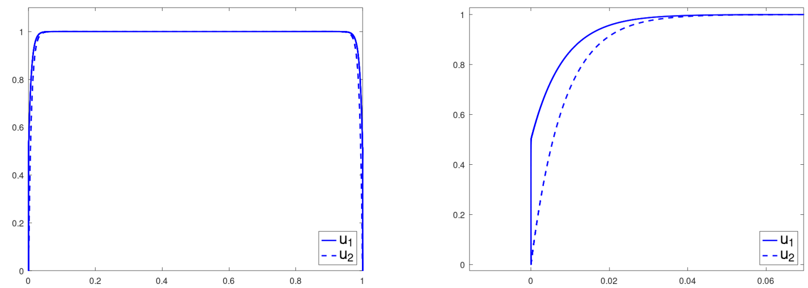

FIGURE 3. Numerical solution of Example 5.2 with $\varepsilon = 10^{-6}$ and $\mu = 10^{-2}$

h	$\varepsilon = 10^{-6}, \mu = 10^{-2}$				$\varepsilon = 10^{-9}, \mu = 10^{-3}$			
	N_h	$\lvert\lvert\lvert\cdot\rvert\rvert\rvert$-error	r_h	r_N	N_h	$\lvert\lvert\lvert\cdot\rvert\rvert\rvert$-error	r_h	r_N
0.32	144	9.0274e-02	-	-	216	9.0090e-02	-	-
0.16	330	4.8701e-02	0.89	0.74	498	4.8628e-02	0.89	0.74
0.08	684	2.5449e-02	0.94	0.89	1036	2.5421e-02	0.94	0.89
0.04	1376	1.3041e-02	0.96	0.96	2082	1.3030e-02	0.96	0.96
0.02	2742	6.6214e-03	0.98	0.98	4148	6.6165e-03	0.98	0.98
0.01	5456	3.3691e-03	0.98	0.98	8252	3.3669e-03	0.97	0.98

TABLE 3. Numerical errors and rates of convergence in balanced norms for Example 5.2

Table 4 shows the numerical errors obtained for the same problem when $\varepsilon = 10^{-9}$ and μ varies between 10^{-9} and 10^{-1}. Since ε is fixed, in all cases, the same ε-graded mesh is used, with $h = 0.1$. We can see that errors remain almost unchanged for all values of μ.

5.3. **Third Example: Variable coefficients, $\varepsilon \neq \mu$.** As a possible line for further research, we deal here with an example which is not covered by the theory of this manuscript. We consider a system with the same matrix of Example 5.1, but with different parameters ε and μ.

We see that the two cases considered in Table 5 show the same orders of convergence of those given in Theorems 4.1 and 4.2.

6. CONCLUSIONS

We have considered the convergence, in a balanced norm, of the linear finite element approximation with graded meshes of a singularly perturbed system of two ordinary differential reaction–diffusion equations. First we have analyzed the case of variable coefficients with the same parameter in both equations and then, we have considered the

μ	$\|\|\|\cdot\|\|\|$-error
1e-01	2.9421e-02
1e-02	3.0373e-02
1e-03	3.1409e-02
1e-04	3.2519e-02
1e-05	3.3707e-02
1e-06	3.4977e-02
1e-07	3.6364e-02
1e-08	3.8466e-02
1e-09	4.4002e-02

TABLE 4. Numerical errors for Example 5.2 with $\varepsilon = 10^{-9}$ and different values of μ. The graded mesh used is the one designed for $\varepsilon = 10^{-9}$ and $h = 0.1$

h	$\varepsilon = 10^{-6}, \mu = 10^{-2}$				$\varepsilon = 10^{-9}, \mu = 10^{-3}$			
	N_h	$\|\|\|\cdot\|\|\|$-error	r_h	r_N	N_h	$\|\|\|\cdot\|\|\|$-error	r_h	r_N
0.32	144	2.7992e-02	-	-	216	2.7737e-02	-	-
0.16	330	1.5173e-02	0.88	0.74	498	1.5037e-02	0.88	0.73
0.08	684	7.9543e-03	0.93	0.89	1036	7.8838e-03	0.93	0.88
0.04	1376	4.0839e-03	0.96	0.95	2082	4.0480e-03	0.96	0.96
0.02	2742	2.0756e-03	0.98	0.98	4148	2.0575e-03	0.98	0.98
0.01	5456	1.0567e-03	0.97	0.98	8252	1.0475e-03	0.97	0.98

TABLE 5. Numerical errors for Example 5.3

case of different small parameters multiplying the second-order derivatives but assuming constant coefficients in both equations. In both cases, almost optimal error estimates with respect to the number of degrees of freedom were proved when appropriate graded meshes are used. Those estimates are robust with respect to the singular perturbation. The goal of our approach is that the proposed graded meshes depend only on the smallest parameter of the system. We also explain that our techniques can be easily extended to systems of more than two equations.

Key tools to obtain our results are H^1 stability estimates for distinct L^2-projections on the finite element space which hold on the (non-quasiuniform) graded meshes. We also include an example of variable coefficients and different perturbation parameters, even though it is not covered by our theory (the projections defined in (26) can not be applied). This issue is subject of future research.

On the other hand, a preliminary analysis shows that graded meshes for higher order approximations require too small elements near the boundary, and therefore the design of practical graded meshes still need further investigation.

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Declarations

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Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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