Anisotropic error estimates for an interpolant defined via moments

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Abstract

An interpolant defined via moments is investigated for triangles, quadrilaterals, tetrahedra, and hexahedra and arbitrarily high polynomial degree. The elements are allowed to have diameters with different asymptotic behavior in different space directions. Anisotropic interpolation error estimates are proved.

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1 Introduction

In this short note we apply the approach for proving anisotropic interpolation error estimates as developed in [1, 2], to a two- or three-dimensional $\mathcal{P}_k$- and $\mathcal{Q}_k$-interpolant defined via moments. To our knowledge, this interpolant is first introduced by Girault and Raviart [5] for triangles and quadrilaterals. The variant with quadrilaterals is used by Lin, Yan and Zhou [6] and Zhou and Li [9] who derive superconvergence results. Stynes and Tobiska [8] derive anisotropic interpolation error estimates for rectangles which will be reproduced as a special case and under weaker assumptions here. Mao and Shi [7] prove error estimates for anisotropic triangles.

The three-dimensional version of this interpolant is used by Buffa, Costabel and Dauge in the paper [4]; anisotropic interpolation error estimates are needed to prove Assumption 4 of this paper. Apel and Matthies [3] estimated the consistency error of non-conforming discretizations of arbitrary order for the Stokes problem where they used the error estimates for this interpolant in the case of rectangles.

Interpolation error estimates are usually proved first for a reference element. In view of [1, Subsection 2.1.2] we will prove in Section 2 the following lemma for several reference elements $T \subset \mathbb{R}^d$, $d = 2, 3$, which will be specified in Section 2.

Lemma 1 *Consider an interpolation operator $I_T : W^{\ell,p}(T) \to \mathcal{P}_T$ where $\mathcal{P}_T \supset \mathcal{P}_k$ is a finite-dimensional space and $\mathcal{P}_k$ is the space of polynomials of order less than or equal to k. Let γ be a multi-index with $|\gamma| = 1$ and $u \in C(\overline{T})$ be a function with $D^\gamma u \in W^{\ell-1,p}$ where $\ell \in \mathbb{N}$ and $p \in [1, \infty]$ shall be such that $2 \le \ell \le k + 1$ and*

$$p > 2 \quad \textit{if } \ell = 2 \textit{ and } d = 3. \tag{1}$$

Fix $q \in [1, \infty]$ such that $W^{\ell-1,p}(T) \subset L^q(T)$. Then the estimate

$$\|D^\gamma(u - I_T u)\|_{L^q(T)} \le C |D^\gamma u|_{W^{\ell-1,p}(T)} \tag{2}$$

holds.

By using the arguments from [1, Chapter 2] we can conclude anisotropic interpolation error estimates of the type

$$|u - I_T u|_{W^{1,q}(T)} \le C |T|^{1/q-1/p} \sum_{|\alpha|=\ell-1} h_T^\alpha |D^\alpha u|_{W^{1,p}(T)} \tag{3}$$

for several types of affine and subparametric elements T with size parameters $h_{1,T}, \dots, h_{d,T}$. This will be surveyed in Section 3.

In this paper we use standard multi-index notation $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_1, \dots, \alpha_d \ge 0$, $|\alpha| = \sum_{i=1}^d \alpha_i$, $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, and $h_T^\alpha = h_{1,T}^{\alpha_1} \cdots h_{d,T}^{\alpha_d}$. As usual, C is a positive constant which is independent of the mesh parameters, i.e., independent of the element sizes and the aspect ratio. Although we investigate families of arbitrary polynomial degree, we focus on an h-version of the finite element method, this means that the dependence of the constants on the polynomial degree is not elaborated.

2 Proof on the reference element

2.1 Background

For the proof of Lemma 1 we will use Lemma 2.2 of [1], see also [2, Lemma 3] for a more general version. We repeat it here using an adapted notation.

Lemma 2 *Let T be a reference element and $I_T : C(\overline{T}) \to \mathcal{P}_T$ be a linear operator. Fix $\ell \in \mathbb{N}$ and $p, q \in [1, \infty]$ such that $1 \le \ell \le k+1$ and $W^{\ell-1,p}(T) \hookrightarrow L^q(T)$. Consider a multi-index γ with $|\gamma| = 1$ and assume that there is a set $\mathcal{F}$ of linear functionals such that*

$$
\begin{aligned}
&\dim \mathcal{F} = \dim D^\gamma \mathcal{P}_T && (4)\\
&F \in \left(W^{\ell-1,p}(T)\right)' \quad \forall F \in \mathcal{F}, && (5)\\
&F(D^\gamma(u - I_T u)) = 0 \quad \forall F \in \mathcal{F}, \quad \forall u \in C(\overline{T}) : D^\gamma u \in W^{\ell-1,p}(T), && (6)\\
&w \in \mathcal{P}_T \quad \text{and} \quad F(D^\gamma w) = 0 \quad \forall F \in \mathcal{F} \quad \Longrightarrow \quad D^\gamma w = 0. && (7)
\end{aligned}
$$

Then the error can be estimated for all $u \in C(\overline{T})$ with $D^\gamma u \in W^{\ell-1,p}(T)$ by

$$\|D^\gamma(u - I_T u)\|_{L^q(T)} \le C |D^\gamma u|_{W^{\ell-1,p}(T)}. \tag{8}$$

According to this lemma, the task is to define the functionals and to prove the properties (4)–(7).

2.2 Triangles

Let T be the usual reference element with the nodes $(0,0)$, $(1,0)$, and $(0,1)$. For a function $u \in C(\overline{T})$ we define the interpolant $I_T u \in \mathcal{P}_T = \mathcal{P}_k(T)$, $k \ge 1$, by

$$
\begin{aligned}
(u - I_T u)(V) &= 0 && \forall \text{ vertices } V \text{ of } T, && (9)\\
\int_E (u - I_T u) q &= 0 && \forall \text{ edges } E \subset \partial T, \quad \forall q \in \mathcal{P}_{k-2}(E), \text{ if } k \ge 2, && (10)\\
\int_T (u - I_T u) q &= 0 && \forall q \in \mathcal{P}_{k-3}(T), \text{ if } k \ge 3. && (11)
\end{aligned}
$$

To show that this interpolant is well defined it suffices to show for $u \equiv 0$ that $I_T u \equiv 0$ is the unique solution of the interpolation conditions. Let E be an arbitrary edge of triangle T. Due to (9) the interpolant vanishes at the vertices of E such that the restriction if $I_T u$ on E can be written as $\lambda_0 \lambda_1 p$ with $p \in \mathcal{P}_{k-2}(E)$, where λ_0 and λ_1 are barycentric coordinates of E. Using (10) leads us to

$$\int_E \lambda_0 \lambda_1 p q = 0 \qquad \forall q \in \mathcal{P}_{k-2}(E)$$

and thus $p \equiv 0$ and therefore $I_T u \equiv 0$ on $\partial\Omega$. Therefore we can write $I_T u = \lambda_0 \lambda_1 \lambda_2 p$ with $p \in \mathcal{P}_{k-3}(T)$ and obtain from (11)

$$\int_T \lambda_0 \lambda_1 \lambda_3 p q = 0 \qquad \forall q \in \mathcal{P}_{k-3}(T),$$

thus $p \equiv 0$ and, consequently, the desired result $I_T u = 0$.

We will now prove Lemma 1 for $\gamma = (1,0)$. Therefore E is in the following the edge of T in x_1-direction. The case $\gamma = (0,1)$ can proved by analogy where the edge in x_2-direction is used. In view of Lemma 2 we define the set $\mathcal{F}$ to include the following functionals

$$v \quad \mapsto \quad \int_E vq, \quad q \in \mathcal{P}_{k-1}(E), \tag{12}$$

$$v \quad \mapsto \quad \int_T vq, \quad q \in \mathcal{P}_{k-2}(T). \tag{13}$$

This notation is to be understood that for the definition of $\mathcal{F}$ the function q in (12) and (13) varies only in a basis of $\mathcal{P}_{k-1}(E)$ and $q \in \mathcal{P}_{k-2}(T)$, respectively. Hence we can show condition (4) by the calculation

$$\dim \mathcal{P}_{k-1}(E) + \dim \mathcal{P}_{k-2}(T) = \binom{k}{1} + \binom{k}{2} = \binom{k+1}{2} = \dim \mathcal{P}_{k-1}(T).$$

These functionals are well defined for $v \in L^1(E) \hookrightarrow W^{1,p}(T)$, $p \geq 1$. Thus we obtain (5) for $\ell \geq 2$ as assumed in Lemma 1.

To prove (6) we integrate partially and use the definition (9)–(11) of I_T:

$$\begin{aligned} \int_E \partial_1(u - I_T u)q &= (u - I_T u)q|_{\partial E} - \int_E (u - I_T u)\partial_1 q = 0 \quad \forall q \in \mathcal{P}_{k-1}(E), \\ \int_T \partial_1(u - I_T u)q &= \int_{\partial T} (u - I_T u)qn_1 - \int_T (u - I_T u)\partial_1 q = 0 \quad \forall q \in \mathcal{P}_{k-2}(T). \end{aligned}$$

It remains to show the independence property (7). Consider a polynomial $w \in \mathcal{P}_k(T)$, i. e. $v := \partial_1 w \in \mathcal{P}_{k-1}(T)$. From

$$\int_E vq = 0 \quad \forall q \in \mathcal{P}_{k-1}(E)$$

we obtain $v \equiv 0$ on E and thus $v = x_2 p$ with $p \in \mathcal{P}_{k-2}(T)$. Consequently we get from

$$\int_T vq = \int_T x_2 pq = 0 \quad \forall q \in \mathcal{P}_{k-2}(T)$$

that $p \equiv 0$ and thus the desired result $v = \partial_1 w \equiv 0$. In this way we have proved all necessary properties of $\mathcal{F}$ and, as a conclusion of Lemma 2, Lemma 1 is proved in the triangular case.

2.3 Tetrahedra

The interpolant $I_T u \in \mathcal{P}_T = \mathcal{P}_k(T)$, $k \geq 1$, is defined in analogy to the triangular case by

$$(u - I_T u)(V) = 0 \qquad \forall \text{ vertices } V \text{ of } T, \tag{14}$$

$$\int_E (u - I_T u)q = 0 \qquad \forall \text{ edges } E \subset \partial T, \quad \forall q \in \mathcal{P}_{k-2}(E), \text{ if } k \geq 2, \tag{15}$$

$$\int_F (u - I_T u)q = 0 \qquad \forall \text{ faces } F \subset \partial T, \quad \forall q \in \mathcal{P}_{k-3}(F), \text{ if } k \geq 3, \tag{16}$$

$$\int_T (u - I_T u)q = 0 \qquad \forall q \in \mathcal{P}_{k-4}(T), \text{ if } k \geq 4. \tag{17}$$

It can be proved as in the triangular case that this interpolant is well defined.

The peculiarity of anisotropic tetrahedral elements is that we have to consider more than one reference element T, see the discussion in [1, Section 2.3]. The common property of the reference elements is that they have three edges of unit length and parallel to the coordinate axes. We prove here Lemma 1 in the case $\gamma = (1,0,0)$ and denote therefore by E the edge of T which is parallel to the x_1-axis. Moreover, let $\mathcal{S}_F$ be the set of the two faces of T that share the edge E and note that these two faces are also parallel to the x_1-axis.

In full analogy to the triangular case we define the functionals

$$\begin{aligned} v &\mapsto \int_E vq, \quad q \in \mathcal{P}_{k-1}(E), & (18)\\ v &\mapsto \int_F vq, \quad q \in \mathcal{P}_{k-2}(F), \quad F \in \mathcal{S}_F, & (19)\\ v &\mapsto \int_T vq, \quad q \in \mathcal{P}_{k-3}(T). & (20)\end{aligned}$$

The number of functionals satisfies (4) since

$$\binom{k}{1} + 2\cdot\binom{k}{2} + \binom{k}{3} = \binom{k+2}{3} = \dim \mathcal{P}_{k-1}(T).$$

These functionals are well defined for $v \in L^1(E) \hookrightarrow W^{\ell-1,p}(T)$ for which we need the condition (1). The interpolation property (6) is proved analogously to the triangular case by partial integration and the definition (14)–(17) of I_T.

Let us finally sketch the proof of the independence property (7) For $v := \partial_1 w \in \mathcal{P}_{k-1}(T)$ we conclude $v \equiv 0$ on E from the vanishing functionals (18). With the same argument as in the proof of the triangular case we find that $v \equiv 0$ on the two faces $S_1, S_2 \in \mathcal{S}_F$. Let now λ_1 and λ_2 be the barycentric coordinates of T that vanish on S_1 and S_2, respectively, such that we can write $v = \lambda_1\lambda_2 p$ with $p \in \mathcal{P}_{k-3}(T)$. With

$$\int_T vq = \int_T \lambda_1\lambda_2 pq = 0 \quad \forall q \in \mathcal{P}_{k-3}(T)$$

we get $p \equiv 0$ and thus $v := \partial_1 w \equiv 0$.

2.4 Quadrilaterals

Let $T = (0,1)^2$ be the reference element and $\mathcal{Q}_k(T)$ be the space of polynomials of order less than or equal to k in each variable. For later use we introduce also the space $\mathcal{Q}_{k,m}(T)$ of polynomials of order less than or equal to k in the first variable and of order less than or equal to m in the second variable. For a function $u \in C(\overline{T})$ we define the interpolant $I_T u \in \mathcal{P}_T = \mathcal{Q}_k(T)$, $k \geq 1$, by

$$\begin{aligned} (u - I_T u)(V) &= 0 \qquad \forall \text{ vertices } V \text{ of } T, & (21)\\ \int_E (u - I_T u)q &= 0 \qquad \forall \text{ edges } E \subset \partial T, \quad \forall q \in \mathcal{P}_{k-2}(E), \text{ if } k \geq 2, & (22)\\ \int_T (u - I_T u)q &= 0 \qquad \forall q \in \mathcal{Q}_{k-2}(T), \text{ if } k \geq 2. & (23)\end{aligned}$$

In order to prove that the interpolant is well defined we consider $u \equiv 0$ as in the triangular case and conclude from conditions (21) and (22) that $I_T u \equiv 0$ on ∂T. We can write $I_T u = bp$ with $p \in \mathcal{Q}_{k-2}(T)$ and a bubble function $b \in \mathcal{Q}_2$ that vanishes on ∂T. Hence the condition (23) becomes

$$\int_T bpq = 0 \qquad \forall q \in \mathcal{Q}_{k-2}(T).$$

Since b does not change sign in T we conclude $p \equiv 0$ and, consequently, $I_T u = 0$.

The proof of Lemma 1 is also similar to the triangular case. Care must be taken about the polynomial spaces and its dimensions. First note that $\partial_1 \mathcal{Q}_k = \mathcal{Q}_{k-1,k}$. Let $\mathcal{S}_E$ be the set of the two edges of T which are parallel to the x_1-axis. Consider the functionals

$$\begin{aligned} v &\mapsto \int_E vq, \quad q \in \mathcal{P}_{k-1}(E), \quad E \in \mathcal{S}_E, & (24)\\ v &\mapsto \int_T vq, \quad q \in \mathcal{Q}_{k-1,k-2}(T). & (25)\end{aligned}$$

The number of functionals satisfies (4) since

$$2 \cdot \dim \mathcal{P}_{k-1}(E) + \dim \mathcal{Q}_{k-1,k-2}(T) = 2k + k(k-1) = k(k+1) = \dim \mathcal{Q}_{k-1,k}(T).$$

These functionals are well defined for $v \in L^1(E) \hookrightarrow W^{\ell-1,p}(T)$ with $\ell \geq 2$. The interpolation property (6) is proved again by partial integration and the definition (21)–(23) of I_T. Here we need in particular that $\partial_1 q \in \mathcal{Q}_{k-2}(T)$ for $q \in \mathcal{Q}_{k-1,k-2}(T)$.

In order to prove the independence consider $v := \partial_1 w \in \mathcal{Q}_{k-1,k}(T)$. From (24) we obtain that $v \equiv 0$ on the two edges $E \in \mathcal{S}_E$, and thus $v = x_2(1-x_2)p$ with $p \in \mathcal{Q}_{k-1,k-2}(T)$. With (25) we obtain $p \equiv 0$ and thus $v := \partial_1 w \equiv 0$.

2.5 Hexahedra

The reference element is $T = (0,1)^3$ and $\mathcal{Q}_k(T)$ is again the space of polynomials of order less than or equal to k in each variable. The space $\mathcal{Q}_{k,m,n}(T)$ is defined in analogy to the quadrilateral case. For a function $u \in C(\overline{T})$ we define the interpolant $I_T u \in \mathcal{P}_T = \mathcal{Q}_k(T)$, $k \geq 1$, by

$$\begin{aligned}
(u - I_T u)(V) &= 0 \qquad \forall \text{ vertices } V \text{ of } T, && (26)\\
\int_E (u - I_T u) q &= 0 \qquad \forall \text{ edges } E \subset \partial T, \quad \forall q \in \mathcal{P}_{k-2}(E), \text{ if } k \geq 2, && (27)\\
\int_F (u - I_T u) q &= 0 \qquad \forall \text{ faces } F \subset \partial T, \quad \forall q \in \mathcal{Q}_{k-2}(F), \text{ if } k \geq 2, && (28)\\
\int_T (u - I_T u) q &= 0 \qquad \forall q \in \mathcal{Q}_{k-2}(T), \text{ if } k \geq 2. && (29)
\end{aligned}$$

It can be proved as in the quadrilateral case that this interpolant is well defined.

In order to prove Lemma 1 we note first that $\partial_1 \mathcal{Q}_k = \mathcal{Q}_{k-1,k,k}$. Let $\mathcal{S}_E$ and $\mathcal{S}_F$ be the set of the four edges/faces of T which are parallel to the x_1-axis. Consider the functionals

$$\begin{aligned}
v &\mapsto \int_E vq, \quad q \in \mathcal{P}_{k-1}(E), \quad E \in \mathcal{S}_E, && (30)\\
v &\mapsto \int_F vq, \quad q \in \mathcal{Q}_{k-1,k-2}(F), \quad F \in \mathcal{S}_F, && (31)\\
v &\mapsto \int_T vq, \quad q \in \mathcal{Q}_{k-1,k-2,k-2}(T), && (32)
\end{aligned}$$

such that

$$\dim \mathcal{F} = 4k + 4k(k-1) + k(k-1)^2 = k(k+1)^2 = \dim \mathcal{Q}_{k-1,k,k}.$$

The boundedness is proved using condition (1) as in the tetrahedral case. The interpolation property follows again by partial integration. Concerning the independence we conclude from (30) that $v = \partial_1 w$ vanishes at the edges of $\mathcal{S}_E$ and, consequently, via (31) also at the faces of $\mathcal{S}_F$ and, finally, with (32) in T.

3 Error estimates in general elements

As in Section 2.2 and Subsections 2.3.1, 2.4.1, and 2.5.1 of [1] we can now consider elements T that are affine images of a reference element from Section 2, and that satisfy the *maximal angle condition* and the *coordinate system condition* as stated in [1]. Using Lemma 1 we see that Theorems 2.1, 2.2, 2.6, 2.9 and Corollaries 2.1, 2.2, 2.3, 2.5 which were proved in [1] for the Lagrangian interpolant, hold also for the interpolant under consideration here. These theorems can be subsumed in the following one.

Theorem 3 *Assume that T is a triangle, tetrahedron, parallelogram or parallelepiped which satisfies the maximal angle condition and the coordinate system condition. Let be $u \in W^{\ell,p}(T)$ where $\ell \in \mathbb{N}$, $\ell \leq k+1$, $p \in [1,\infty]$ satisfy (1). Fix $q \in [1,\infty]$ such that $W^{\ell-1,p}(T) \hookrightarrow L^q(T)$. Then the anisotropic interpolation error estimate*

$$|u - I_T u|_{W^{1,q}(T)} \leq C |T|^{1/q-1/p} \sum_{|\alpha|=\ell-1} h^\alpha \, |D^\alpha u|_{W^{1,p}(T)} \qquad (33)$$

holds.

We could also repeat here the discussion of sharper estimates for rectangular and brick elements in [1] (compare Theorems 2.7 and 2.10 of [1]), or of estimates for functions $u \in W^{k+2}(T)$ with additional regularity (compare Theorems 2.3 and 2.11 of [1]). Moreover, following [1, Subsection 2.3.2] it is also possible to extend the results to the case in which the regularity of u is described in certain classes of weighted Sobolev spaces. — Rather, we elaborate here the case of certain non-affine elements as in Subsections 2.4.3 and 2.5.2 of [1] since these results seem to be of importance and not widely known yet.

The difficulty with non-affine elements is that second order derivatives of the transformation do not vanish. Hence, by the chain rule, the transformation of any mixed derivatives of order ℓ includes also lower order derivatives which lead to lower order terms in the right hand sides of estimates like (33). The remedy is to use sharper estimates on the reference element, see, for example, the discussion in [5, Section A.2], or to restrict the shape of the elements to small perturbations of affine elements.

For our purposes, let us deal with a class of quadrilateral or hexahedral elements T. The coordinates of the vertices of T are perturbations of the vertices of an axiparallel rectangular/brick element $\tilde{T}$ by vectors $a^{(j)} = (a_1^{(j)}, \dots, a_d^{(j)})$, $j = 1, \dots, 2^d$, $d = 2, 3$. We restrict ourselves to the case that the edges of T are straight such that the reference square/cube can be mapped to the element T by a bi-/trilinear coordinate transformation. This class of elements is sometimes called *subparametric* since, contrary to *isoparametric* elements, in general only a subspace of $\mathcal{P}_T$ is used for the transformation.

For the perturbations we demand the existence of non-negative constants $a_0, \dots, a_d$ with

$$|a_i^{(j)}| \le a_i h_{d,T}, \quad i = 1, \dots, d, \quad j = 1, \dots, 2^d, \tag{34}$$

$$\frac{1}{2} - h_{d,T} \sum_{i=1}^{d} \frac{a_i}{h_{i,T}} \ge a_0 > 0. \tag{35}$$

Condition (34) ensures that the perturbation is only of order $h_{d,T}$ and hence the element T is anisotropic with essentially the same stretching direction and aspect ratio as $\tilde{T}$. The other condition, (35), ensures that the mapping from the reference element to the element T is invertible; see also Remarks 2.10 and 2.11 of [1] for further discussion. In particular, trapezoidal elements are included where the length of the parallel sides is of order $h_{1,T}$ and the length of the other two sides is of order $h_{2,T}$.

Theorem 4 *Consider a quadrilateral or hexahedral element T from the class just described. As in Theorem 3, let be $u \in W^{\ell,p}(T)$ where $\ell \in \mathbb{N}$, $\ell \le k+1$, $p \in [1,\infty]$ satisfy (1). Fix $q \in [1,\infty]$ such that $W^{\ell-1,p}(T) \hookrightarrow L^q(T)$. Then the anisotropic interpolation error estimate (33) holds.*

The proof of this theorem differs not much from that for Lagrangian elements. Therefore we refer to Subsections 2.4.3 and 2.5.2 of [1] and sketch here only the main ideas. The first ingredient is a sharper estimate on the reference element. In view of Lemma 2.15 and Remark 2.8 of [1], we have proved for the unit square/cube T in Subsections 2.4 and 2.5 the estimate

$$\|D^\gamma (u - I_T u)\|_{L^q(T)} \le C [D^\gamma u]_{W^{\ell-1,p}(T)}$$

where $[\,\cdot\,]_{W^{\ell-1,p}(T)}$ denotes the seminorm in $W^{\ell-1,p}(T)$ where only pure derivatives are included. Nonetheless, the non-affine transformation with the restrictions above leads in the first place to a weaker estimate than (33). For simplicity of presentation, we add more terms on the right hand side such that the estimate becomes

$$|u - I_T u|_{W^{1,q}(T)} \le C |T|^{1/q - 1/p} \sum_{|\alpha| \le \ell - 1} h^\alpha \, |D^\alpha u|_{W^{1,p}(T)},$$

compare Lemma 2.17 of [1]. Thereafter, we get rid of the lower order terms by a Deny–Lions type argument having observed that $I_T w = w$ for all $w \in \mathcal{P}_{\ell-1}(T)$, see the proof of Theorem 2.8 in [1] for details.

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