Mathematical and Numerical Analysis of Koiter Elliptic Shells in Normal Compliance Contact

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Abstract

We consider a model for Koiter linear elastic elliptic shells in contact with a deformable obstacle and we study the convergence of the solution of this model towards the solution of the corresponding model for elastic elliptic membrane shells when the small parameter of the model (thickness) tends to zero. Furthermore, we propose a numerical scheme for this kind of contact problems for Koiter shells and show numerical simulations after implementation by using the free software package FreeFem++.

Keywords

Shells, Membrane, Koiter, Contact, Asymptotic Analysis, Numerical Analysis, Elasticity, Deformable Foundation, Normal Compliance, Convergence

1 Introduction

In solid mechanics, shells are three-dimensional structures of small thickness compared to the extension they cover. Such structures are abundant in nature (eggs, snails, turtles, blood vessels, . . .) but also in industry (ship hulls, plane fuselage, roofs, glasses, tires, . . .). One of the reasons for this popularity is because of their ability to sustain applied loads in a very effective way, with the minimum amount of material they require, and the lightness and economy that this represents.

Over the last decades, the mathematical community specialized in solid mechanics has contributed to justifying two-dimensional models for elastic shells already existing in the

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scientific literature (which had been obtained mainly through *a priori* geometric and mechanical considerations and heuristic techniques), and has also obtained other new models. One of the main tools used in this endeavour has been asymptotic analysis methods for problems depending on a small parameter, which can be consulted in Lions[1]. These techniques together with other methods of functional analysis, applied in the context of three-dimensional elastic shells, led to the pioneering works of Ciarlet and collaborators. We can go back to the early years of the decade of the 80s of the last century to find Destuynder's[2,3] early works on the justification of elastic plates and shells models using asymptotic analysis techniques. Throughout the last two decades of the 20th century, the works of Lods[4], Miara[5] and Sánchez-Palencia[6] developed methods that allow justifying models for elastic shells of the elliptic membrane kind, generalized membranes or flexural shell kinds. Also at the end of the century, Mardare's work[7] was published, providing error estimates in the approximation of three-dimensional elastic elliptic membrane shell models using the corresponding two-dimensional limit model. This required additional regularity assumptions not only for data but also in the solution of the two-dimensional model. Such additional regularity was investigated by Genevey[8] also in this period. All of this and much more was compiled in the beautifully written third volume of Ciarlet's Mathematical Elasticity at the beginning of the 21st century[9]. It is worth mentioning here some works on shells with little regularity (see for example,[10–12]). The use of surface meshes instead of three-dimensional meshes is a major advantage of shells. But as a drawback there is the need to deal with the observed numerical locking, as we can learn from the work of Bernadou[13] and Bathe-Chapelle[14].

It is precisely at the beginning of the present century that the development of the mathematical theory of contact receives renewed impulse. Thus, following in the footsteps of seminal works such as that of Signorini or Fichera, there were decisive contributions from Jarusek et al.[15], Haslinger et al.[16], Cocou[17] and Sofonea et al.[18–20] to cite some few references. Also very notable has been the contribution of Migorski et al.[21] among many other references. The modeling of a large part of the phenomena associated with contact is due to Shillor et al.[22] and the development of numerical analysis was initiated by Kikuchi and Oden[23] has been decisively pushed forward by Han et al.[24].

In the middle of the second decade of this century, we observed that there was a lack of models of shells in contact with an obstacle, despite the wide range of applications where both shells and contact come together. The only reference we were aware of was due to Leger and Miara[25], in their study of shallow shells in unilateral contact with an obstacle, where the curvature is so small that the treatment can resemble that of a plate. It is in this context that we began a research program that has led us to formulate and justify models for elastic elliptic membrane shells in contact with a rigid obstacle in Arós[26,27] or with a deformable obstacle in Arós-Cao[28]. For both cases, error estimation results have been obtained, in Cao-Arós[29] and Cao et al.[30], respectively. More recently, thermoelastic shells in dynamic contact with an obstacle modeled with a normal damped response condition has been studied in Cao et al.[31], also in cases of bilateral contact with Tresca friction in Arós-Castiñeira-Viaño[32], or friction coupled with other tribologic phenomena such as wear in Arós et el.[33]. In parallel, Piersanti (see for example[34]) has strongly developed the study of elastic shells with unilateral confinement-type conditions.

Besides, in the literature we can find models that do not arise naturally from a formal asymptotic exploration. Among them, it is worth highlighting the Koiter model or the Naghdi model. Specifically, in Ciarlet[9] Chapter 7 we find a discussion that Koiter's two-dimensional model is not only a valid approximation of the three-dimensional model, but probably a better choice than the membrane or flexural shell models alone. From the practical point of view this is a major advantage since one can use always a Koiter model regardless of the geometry or the boundary conditions, knowing that it will be a good approximation of whatever the actual situation is. In this work we propose to start a similar study but under the additional assumption that the Koiter shell can come into contact with a deformable obstacle. We will show that as the small parameter tends to zero, the Koiter model approaches that of the Arós-Cao[28] elastic elliptic membrane model. Furthermore, we propose a numerical scheme and implement it in the framework provided by the free software FreeFem++ (see, for example[35]). Thus, we show simulations in which we observe the effect of reducing thickness of the shell or increasing the rigidity of the obstacle. We also analyse the convergences rates, both when the mesh is refined and when the thickness of the shell is reduced. We refer the reader to[36] Section 6.1 for a numerical validation of the Koiter model in the case without an obstacle. Further results concerning numerical approximations of obstacle problems by means of the Finite Element Method can be found in[37]. In that work, the authors analyze a penalized formulation of the problem describing the deformation of a linearly elastic elliptic membrane shell constrained to remain within a half-space, and employ the Brezis–Sibony iterative scheme to address the resulting nonlinear discrete problem.

The structure of the paper is the following: in Section 2 we shall recall the variational formulation of the three-dimensional contact problem in scaled curvilinear coordinates. Then, in Section 3 we shall recall the two-dimensional limit model obtained for elastic elliptic membrane shells in[28], formulate the corresponding Koiter model for the same boundary conditions and applied forces, study existence and uniqueness of solution, and provide a convergence result as the thickness tends to zero. Next, in Section 4 we formulate a penalized version of the problem for the Koiter shells and provide a convergence result as the penalization parameter tends to zero. Then, in Section 5 we formulate a discrete scheme and provide a priori error estimates. Numerical simulations will be provided and discussed in Section 6. The paper ends with Section 7, devoted to conclusions and future work.

2 The Three-Dimensional Problem for Elastic Shells in Normal Compliance Contact

This section summarizes essential notations and preliminary results. For further details, readers may consult[9] or[27]. From now on, Greek indices take their values in the set $\{1, 2\}$, while Latin indices do it in the set $\{1, 2, 3\}$. Additionally, for the sake of briefness, we might omit the explicit dependence of the unknowns and functions on the space variable, when there is no ambiguity.

Two dimensional domain definition and surface geometry

Consider a domain $\omega \subset \mathbb{R}^2$ with Lipschitz-continuous boundary $\gamma = \partial\omega$. We denote by $\boldsymbol{y} = (y_\alpha)$ a generic point in $\bar{\omega}$, and by ∂_α the partial differentiation with respect to y_α.

We consider a surface defined as $S := \boldsymbol{\theta}(\bar{\omega})$, where $\boldsymbol{\theta} \in \mathcal{C}^1(\bar{\omega}; \mathbb{R}^3)$ is assumed to be an injective mapping such that the vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ form the covariant basis of the tangent plane to the surface S at the point $\hat{\boldsymbol{y}} = \boldsymbol{\theta}(\boldsymbol{y})$. Associated with this covariant basis, we define the contravariant basis $\boldsymbol{a}^\alpha(\boldsymbol{y})$ given by the relations $\boldsymbol{a}^\alpha(\boldsymbol{y}) \cdot \boldsymbol{a}_\beta(\boldsymbol{y}) = \delta^\alpha_\beta$.

These bases allow us to characterize several fundamental geometric quantities of the surface S defined at $\boldsymbol{y} \in \omega$:

$$\begin{aligned}
&\text{Unit normal vector:} && \boldsymbol{a}_3 = \boldsymbol{a}^3 := \tfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|},\\
&\text{Metric tensor:} && a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \text{ (covariant components)}\\
& && a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta \text{ (contravariant components)},\\
&\text{Curvature tensor:} && b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha \text{ (covariant components)} ,\\
& && b^\beta_\alpha := a^{\beta\sigma} \cdot b_{\sigma\alpha} \text{ (mixed components)} ,\\
&\text{Christoffel symbols:} && \Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha,\\
&\text{Area element:} && \sqrt{a}dy, \text{ where } a := \det(a_{\alpha\beta}).
\end{aligned}$$

Notice that above and in what follows a centered dot, $\cdot$, denotes scalar product while a wedge symbol, $\wedge$, denotes vector product in $\mathbb{R}^3$.

Three dimensional domain definition and shell geometry

We want to describe a deformable body consisting of all points within a given distance $\varepsilon > 0$ from the surface S. We start by denoting $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ a three-dimensional cylindrical domain, and $\Gamma^\varepsilon = \partial\Omega^\varepsilon$ its boundary. We consider a partition of Γ^ε in the following parts:

$$\Gamma^\varepsilon_+ := \omega \times \{\varepsilon\}, \qquad \Gamma^\varepsilon_C := \omega \times \{-\varepsilon\}, \qquad \Gamma^\varepsilon_L := \gamma \times (-\varepsilon, \varepsilon).$$

Besides, homogeneous Dirichlet conditions are defined on a subset of Γ^ε_L denoted Γ^ε_0 and defined by

$$\Gamma^\varepsilon_0 := \gamma_0 \times (-\varepsilon, \varepsilon),$$

where $\gamma_0 \subseteq \gamma$ has nonzero measure. Moreover, we use $\boldsymbol{n}^\varepsilon$ to denote the unit outward normal vector on Γ^ε. We denote by $\boldsymbol{x}^\varepsilon = (x^\varepsilon_i)$ a generic point of $\bar{\Omega}^\varepsilon$ and by ∂^ε_i the partial derivative with respect to x^ε_i. It's clear that $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$.

The three-dimensional body is then parametrized as the image of the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ given by:

$$\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(\boldsymbol{y}) + x^\varepsilon_3 \boldsymbol{a}_3(\boldsymbol{y}) \; \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, x^\varepsilon_3) = (y_1, y_2, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon. \tag{1}$$

In order to avoid self-intersections in the reference configuration we suppose that $\varepsilon > 0$ is small enough so that, under the injectivity and smoothness assumptions on $\boldsymbol{\theta}$, the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is also injective and the vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon) := \partial^\varepsilon_i \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent (see[9] Th. 3.1-1). These vectors form the covariant basis at each point $\hat{\boldsymbol{x}}^\varepsilon := \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. Notice that a hat above makes reference to cartesian coordinates while an ε superindex indicates dependence on the small parameter. The corresponding contravariant basis $\boldsymbol{g}^{i,\varepsilon}$ is defined by the relation $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^\varepsilon_j = \delta^i_j$.

We define the following geometric elements associated to the deformable body $\hat{\Omega}^\varepsilon := \boldsymbol{\Theta}(\Omega^\varepsilon)$:

Unit outward normal vector (on $\hat{\boldsymbol{x}}^\varepsilon \in \boldsymbol{\Theta}(\Gamma^\varepsilon)$): $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = \dfrac{\mathrm{Cof}\,(\nabla\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon))\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)}{|\mathrm{Cof}\,(\nabla\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon))\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)|}$,

Metric tensor: $g^\varepsilon_{ij} := \boldsymbol{g}^\varepsilon_i \cdot \boldsymbol{g}^\varepsilon_j$ (covariant components)
$g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}$ (contravariant components),

Christoffel symbols: $\Gamma^{p,\varepsilon}_{ij} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial^\varepsilon_i \boldsymbol{g}^\varepsilon_j$,

Volume element (in $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$): $\sqrt{g^\varepsilon}dx^\varepsilon$, where $g^\varepsilon := \det(g^\varepsilon_{ij})$,

Surface element (in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$): $\sqrt{g^\varepsilon}d\Gamma^\varepsilon$,

where $\mathrm{Cof}\,(A)$ denotes the cofactor matrix of a given regular matrix A. It's straightforward from (1) that $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = -\boldsymbol{g}_3(\boldsymbol{x}^\varepsilon) = -\boldsymbol{a}_3(\boldsymbol{y})$, where $\boldsymbol{x}^\varepsilon = (\boldsymbol{y}, -\varepsilon) \in \Gamma^\varepsilon_C$. On the other hand, given a field $\hat{\boldsymbol{v}}^\varepsilon : \boldsymbol{\Theta}(\bar{\Omega}^\varepsilon) \to \mathbb{R}^3$, we define its covariant curvilinear coordinates (v^ε_i) in $\bar{\Omega}^\varepsilon$ as the set of scalars which yield the equality $v^\varepsilon_i(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon) = \hat{\boldsymbol{v}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = \hat{v}^\varepsilon_i(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}^i$ with $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$, where the cartesian basis is denoted by $\{\hat{\boldsymbol{e}}^i\}_{i=1}^3$. Therefore, it can be shown that on Γ^ε_C the normal component of the field verifies $\hat{v}_n = -v^\varepsilon_3$.

We consider the domain $\hat{\Omega}^\varepsilon$ as a natural configuration of an elastic, homogeneous, and isotropic material, characterized by Lamé coefficients $\lambda \geq 0$ and $\mu > 0$, both independent of ε. The displacement field is assumed to vanish on a portion of the lateral boundary, $\hat{\Gamma}^\varepsilon_0 := \boldsymbol{\Theta}(\Gamma^\varepsilon_0)$.

Variational formulation

The physical setting of the contact problem is as follows. First, the shell is under the influence of body forces $\hat{\boldsymbol{f}}^\varepsilon = (\hat{f}^{i,\varepsilon})$ within $\hat{\Omega}^\varepsilon$ and surface tractions $\hat{\boldsymbol{h}}^\varepsilon = (\hat{h}^{i,\varepsilon})$ applied on the upper face $\hat{\Gamma}^\varepsilon_+ := \boldsymbol{\Theta}(\Gamma^\varepsilon_+)$. Additionally, the shell may come into contact with a deformable foundation initially separated by a known gap $\hat{s}^\varepsilon$, measured along the outward unit normal $\hat{\boldsymbol{n}}^\varepsilon$ on the potential contact boundary $\hat{\Gamma}^\varepsilon_C := \boldsymbol{\Theta}(\Gamma^\varepsilon_C)$.

Assuming frictionless contact, the normal interaction between the shell and the foundation can be modeled via a normal compliance law, in the following form:

$$\hat{\sigma}^\varepsilon_n = -k^\varepsilon_n(\hat{u}^\varepsilon_n - \hat{s}^\varepsilon)_+, \qquad \hat{\boldsymbol{\sigma}}^\varepsilon_\tau = \boldsymbol{0}, \tag{2}$$

where $\hat{u}^\varepsilon_n$ is the normal component of the displacement, $\hat{\sigma}^\varepsilon_n$ represents the normal stress and $\hat{\boldsymbol{\sigma}}^\varepsilon_\tau$ denotes the tangential stress (see for example[24] for more details). Also, $k^\varepsilon_n > 0$ measures the rigidity of the foundation and $r_+ := \max\{r, 0\}$ is the positive part function. More generally, the contact response may be governed by a nonlinear compliance function $\hat{p}^\varepsilon : \mathbb{R} \to \mathbb{R}_+$ satisfying:

$$\hat{\sigma}^\varepsilon_n = -\hat{p}^\varepsilon(\hat{u}^\varepsilon_n - \hat{s}^\varepsilon) \quad \text{on } \hat{\Gamma}^\varepsilon_C,$$

where $\hat{p}^\varepsilon$ satisfies the following conditions:

$$\begin{aligned}
&\text{(a)} \quad \hat{p}^\varepsilon(r) = 0 \quad \text{for } r \leq 0,\\
&\text{(b)} \quad \exists\, L^\varepsilon_p > 0 \text{ such that } |\hat{p}^\varepsilon(r_1) - \hat{p}^\varepsilon(r_2)| \leq L^\varepsilon_p|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R},\\
&\text{(c)} \quad (\hat{p}^\varepsilon(r_1) - \hat{p}^\varepsilon(r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}.
\end{aligned} \tag{3}$$

The case described in (2) corresponds to the choice:

$$\hat{p}^\varepsilon(r) = k_n^\varepsilon r_+. \tag{4}$$

Notice that for this case it is straightforward to identity $L_p^\varepsilon = k_n^\varepsilon$ in (3) (b).

It is straightforward to derive the variational formulation of the frictionless normal compliance contact problem between the three-dimensional solid and the deformable foundation, in cartesian coordinates, which is the following (see[28]):

Problem 2.1. *Find* $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}_i^\varepsilon) : \hat{\Omega}^\varepsilon \to \mathbb{R}^3$ *such that,*

$$\begin{aligned} &\hat{\boldsymbol{u}}^\varepsilon \in V(\hat{\Omega}^\varepsilon) := \{\hat{\boldsymbol{v}}^\varepsilon = (\hat{v}_i^\varepsilon) \in [H^1(\hat{\Omega}^\varepsilon)]^3; \hat{\boldsymbol{v}}^\varepsilon = \boldsymbol{0} \text{ on } \hat{\Gamma}_0^\varepsilon\}, \\ &\int_{\hat{\Omega}^\varepsilon} \hat{A}^{ijkl,\varepsilon} \hat{e}_{kl}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon) \hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon) d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_C^\varepsilon} \hat{p}^\varepsilon(\hat{u}_n^\varepsilon - \hat{s}^\varepsilon)\hat{v}_n^\varepsilon \, d\hat{\Gamma}^\varepsilon \\ &\quad = \int_{\hat{\Omega}^\varepsilon} \hat{f}^{i,\varepsilon} \hat{v}_i^\varepsilon \, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon} \hat{h}^{i,\varepsilon} \hat{v}_i^\varepsilon \, d\hat{\Gamma}^\varepsilon \quad \forall \hat{\boldsymbol{v}}^\varepsilon \in V(\hat{\Omega}^\varepsilon), \end{aligned}$$

where $\hat{A}^{ijkl,\varepsilon} = \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_j \hat{v}_i^\varepsilon + \hat{\partial}_i \hat{v}_j^\varepsilon)$ denote the components of the elasticity fourth-order tensor and the linearized strain tensor, respectively. Here δ^{ij} represents the Kronecker's symbol and $\hat{\partial}_i$ the partial derivative with respect to $\hat{x}_i$.

Next, we define the correspondences in curvilinear coordinates for the functions and unknowns in the cartesian formulation of Problem 2.1:

Applied forces densities: $\hat{f}^{i,\varepsilon}(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}_i =: f^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}_i^\varepsilon(\boldsymbol{x}^\varepsilon)$

$\hat{h}^{i,\varepsilon}(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}_i d\hat{\Gamma}^\varepsilon =: h^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}_i^\varepsilon(\boldsymbol{x}^\varepsilon)\sqrt{g^\varepsilon(\boldsymbol{x}^\varepsilon)}d\Gamma^\varepsilon,$

Displacements field: $\hat{\boldsymbol{u}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = \hat{u}_i^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}^i =: u_i^\varepsilon(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon),$

Gap function: $\hat{s}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) =: s^\varepsilon(\boldsymbol{x}^\varepsilon),$

Normal compliance function: $p^\varepsilon(r^\varepsilon) := \hat{p}^\varepsilon(\hat{r}^\varepsilon) \quad \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, -\varepsilon),\ \boldsymbol{y} \in \omega,$

with $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. We also define the following space,

$$V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3; \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\},$$

which is a real Hilbert space with the induced inner product of $[H^1(\Omega^\varepsilon)]^3$. We denote the corresponding norm by $||\cdot||_{1,\Omega^\varepsilon}$. Following similar arguments as those used in[9] for problems without contact conditions on the boundary, we can derive the following variational problem in the curvilinear coordinates framework (see[28]):

Problem 2.2. *Find* $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \Omega^\varepsilon \to \mathbb{R}^3$ *such that,*

$$\begin{aligned} &\boldsymbol{u}^\varepsilon \in V(\Omega^\varepsilon), \quad \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k||l}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i||j}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon - \int_{\Gamma_C^\varepsilon} p^\varepsilon(-u_3^\varepsilon - s^\varepsilon) v_3^\varepsilon \sqrt{g^\varepsilon} d\Gamma^\varepsilon \\ &= \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Gamma_+^\varepsilon} h^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon} d\Gamma^\varepsilon \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon), \end{aligned}$$

where we employ the following notations:

$$\text{Elasticity tensor:} \quad A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon})$$

$$\text{Strain tensor:} \quad e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon}) := \frac{1}{2}(\partial^{\varepsilon}_j v^{\varepsilon}_i + \partial^{\varepsilon}_i v^{\varepsilon}_j) - \Gamma^{p,\varepsilon}_{ij} v^{\varepsilon}_p,$$

Besides, the following properties hold:

$$A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon} \in \mathcal{C}^1(\bar{\Omega}^{\varepsilon}), \qquad e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon}) = e^{\varepsilon}_{j||i}(\boldsymbol{v}^{\varepsilon}) \in L^2(\Omega^{\varepsilon})$$

$$\Gamma^{3,\varepsilon}_{\alpha 3} = \Gamma^{p,\varepsilon}_{33} = 0 \text{ in } \bar{\Omega}^{\varepsilon}, \quad A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = 0 \text{ in } \bar{\Omega}^{\varepsilon},$$

Following[9] Theorem 1.8-1, the choice of a sufficiently small $\varepsilon > 0$ and the definition of the fourth order tensor allows us to conclude the existence of a constant $C_e > 0$, independent of ε, such that,

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl,\varepsilon}(\boldsymbol{x}^{\varepsilon}) t_{kl} t_{ij}, \tag{5}$$

for all $\boldsymbol{x}^{\varepsilon} \in \bar{\Omega}^{\varepsilon}$ and all $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^2$ (the space of real symmetric square matrices of two columns).

We can prove the existence and uniqueness of $\boldsymbol{u}^{\varepsilon} \in V(\Omega^{\varepsilon})$, solution of Problem 2.2, applying the theory of elliptic variational equations (see, for example[24] Cor. 4.4). In particular, we employ (5), the properties of the normal compliance function (3) and a suitable Korn inequality (see for example[9] Th. 1.7-4).

Strong formulation

Additionally, if the functions involved have sufficient regularity, from Problem 2.2 we can deduce the following strong formulation:

Problem 2.3. *Find* $\boldsymbol{u}^{\varepsilon} = (u^{\varepsilon}_i) : \Omega^{\varepsilon} \to \mathbb{R}^3$ *such that,*

$$\begin{aligned}
&-\sigma^{ij,\varepsilon}||_j(\boldsymbol{u}^{\varepsilon}) = f^{i,\varepsilon} \ \textit{in } \Omega^{\varepsilon},\\
&u^{\varepsilon}_i = 0 \ \textit{on } \Gamma^{\varepsilon}_0,\\
&\sigma^{ij,\varepsilon}(\boldsymbol{u}^{\varepsilon}) n^{\varepsilon}_j = h^{i,\varepsilon} \ \textit{on } \Gamma^{\varepsilon}_+,\\
&\sigma^{33,\varepsilon}(\boldsymbol{u}^{\varepsilon}) = -p^{\varepsilon}(-u^{\varepsilon}_3 - s^{\varepsilon}), \ \sigma^{\alpha 3,\varepsilon}(\boldsymbol{u}^{\varepsilon}) = \boldsymbol{0} \quad \textit{on } \Gamma^{\varepsilon}_C,
\end{aligned}$$

where the functions $\sigma^{ij,\varepsilon}(\boldsymbol{u}^{\varepsilon}) := A^{ijkl,\varepsilon} e^{\varepsilon}_{k||l}(\boldsymbol{u}^{\varepsilon})$ *are the contravariant components of the linearized stress tensor field and the functions*

$$\sigma^{ij,\varepsilon}||_k(\boldsymbol{u}^{\varepsilon}) := \partial^{\varepsilon}_k \sigma^{ij,\varepsilon}(\boldsymbol{u}^{\varepsilon}) + \Gamma^{i,\varepsilon}_{pk} \sigma^{pj,\varepsilon}(\boldsymbol{u}^{\varepsilon}) + \Gamma^{j,\varepsilon}_{kq} \sigma^{iq,\varepsilon}(\boldsymbol{u}^{\varepsilon}),$$

denote the first-order covariant derivatives of the stress tensor components.

Scaled domain and variational problem

Now, we aim to reformulate the problem on a reference domain that is independent of the small parameter ε. To this end, we define the three-dimensional domain $\Omega := \omega \times (-1, 1)$ with boundary $\Gamma = \partial\Omega$. As in the case of Ω^ε, we partition the boundary as follows:

$$\Gamma_+ := \omega \times \{1\}, \quad \Gamma_C := \omega \times \{-1\}, \quad \Gamma_L := \gamma \times (-1,1).$$

We also define the subset $\Gamma_0 := \gamma_0 \times (-1,1) \subseteq \Gamma_L$. We denote a generic point in $\bar{\Omega}$ by $\boldsymbol{x} = (x_1, x_2, x_3)$, and ∂_i the partial derivative with respect to x_i. To map this new reference domain onto the original domain, we introduce the bijection map $\pi^\varepsilon : \bar{\Omega} \to \bar{\Omega}^\varepsilon$ defined by

$$\pi^\varepsilon(\boldsymbol{x}) = \boldsymbol{x}^\varepsilon = (x_i^\varepsilon) = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon.$$

Then, we can define scaled versions of the unknown $\boldsymbol{u}^\varepsilon$ and vector fields $\boldsymbol{v}^\varepsilon$ as follows:

$$\begin{aligned}
&\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \to \mathbb{R}^3, \text{ such that } u_i(\varepsilon)(\boldsymbol{x}) := u_i^\varepsilon(\boldsymbol{x}^\varepsilon), \ \forall \boldsymbol{x} \in \bar{\Omega}, \ \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon, \\
&\boldsymbol{v} = (v_i) : \bar{\Omega} \to \mathbb{R}^3, \text{ such that } v_i(\boldsymbol{x}) := v_i^\varepsilon(\boldsymbol{x}^\varepsilon) \ \forall \boldsymbol{x} \in \bar{\Omega}, \ \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon,
\end{aligned}$$

and scaled versions of functions associated with the variational problem 2.2:

$$\Gamma_{ij}^p(\varepsilon)(\boldsymbol{x}) := \Gamma_{ij}^{p,\varepsilon}(\boldsymbol{x}^\varepsilon), \ g(\varepsilon)(\boldsymbol{x}) := g^\varepsilon(\boldsymbol{x}^\varepsilon), \ A^{ijkl}(\varepsilon)(\boldsymbol{x}) := A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon), \forall \boldsymbol{x} \in \bar{\Omega}, \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon.$$

Moreover, it can be shown the existence of $\varepsilon_0 > 0$, and a constant $C_e > 0$ independent of the variables and ε, such that

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl}(\varepsilon)(\boldsymbol{x}) t_{kl} t_{ij}, \tag{6}$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, for all $\boldsymbol{x} \in \bar{\Omega}$ and all $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^2$. Noting that $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$, we introduce the scaled linearized strains $(e_{i||j}(\varepsilon)(\boldsymbol{v})) \in [L^2(\Omega)]^{3\times 3}_{sym}$, which we also denote as $(e_{i||j}(\varepsilon; \boldsymbol{v}))$, defined for all $\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3$ by

$$e_{\alpha||\beta}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma_{\alpha\beta}^p(\varepsilon) v_p, \tag{7}$$

$$e_{\alpha||3}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma_{\alpha 3}^p(\varepsilon) v_p, \tag{8}$$

$$e_{3||3}(\varepsilon; \boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{9}$$

Note that with these definitions it is verified that

$$e_{i||j}^\varepsilon(\boldsymbol{v}^\varepsilon)(\pi^\varepsilon(\boldsymbol{x})) = e_{i||j}(\varepsilon; \boldsymbol{v})(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \Omega.$$

Remark 2.4. The functions $\Gamma_{ij}^p(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ *converge in* $\mathcal{C}^0(\bar{\Omega})$ *when* ε *tends to zero.*

Remark 2.5. For $\varepsilon = 0$ *the functions will be defined with respect to* $\boldsymbol{y} \in \bar{\omega}$. *In this limiting case, it is important to note the singularities in (8) and (9). To avoid ambiguity, we distinguish the three-dimensional Christoffel symbols from the two-dimensional ones associated to* S *by using* $\Gamma_{\alpha\beta}^\sigma(\varepsilon)$ *and* $\Gamma_{\alpha\beta}^\sigma$, *respectively.*

Finally, we define the scaled applied forces $\boldsymbol{f}(\varepsilon) : \Omega \to \mathbb{R}^3$ and $\boldsymbol{h}(\varepsilon) : \Gamma_+ \to \mathbb{R}^3$, by

$$\boldsymbol{f}^\varepsilon = (f^{i,\varepsilon})(\boldsymbol{x}^\varepsilon) =: \boldsymbol{f}(\varepsilon) = (f^i(\varepsilon))(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon,$$
$$\boldsymbol{h}^\varepsilon = (h^{i,\varepsilon})(\boldsymbol{x}^\varepsilon) =: \boldsymbol{h}(\varepsilon) = (h^i(\varepsilon))(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma_+, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Gamma_+^\varepsilon,$$

and the scaled gap $s(\varepsilon) : \Gamma_C \to \mathbb{R}$ and the scaled normal compliance function as follows

$$s(\varepsilon)(\boldsymbol{x}) := s^\varepsilon(\boldsymbol{x}^\varepsilon) \quad \forall \boldsymbol{x} \in \Gamma_C, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Gamma_C^\varepsilon,$$
$$p(\varepsilon)(\boldsymbol{x}, r(\varepsilon)) := p^\varepsilon(\boldsymbol{x}^\varepsilon, r^\varepsilon) \; \forall \boldsymbol{x} = (\boldsymbol{y}, -1), \; \boldsymbol{y} \in \omega, \text{with } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}).$$

This way,

$$u_3^\varepsilon(\boldsymbol{x}^\varepsilon) + s^\varepsilon(\boldsymbol{x}^\varepsilon) \geq 0 \Leftrightarrow u_3(\varepsilon)(\boldsymbol{x}) + s(\varepsilon)(\boldsymbol{x}) \geq 0 \quad \text{with } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Gamma_C^\varepsilon.$$

The scaled variational problem can then be written as follows:

Problem 2.6. *Find* $\boldsymbol{u}(\varepsilon) : \Omega \to \mathbb{R}^3$ *such that,*

$$\boldsymbol{u}(\varepsilon) \in V(\Omega) := \{\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3; \boldsymbol{v} = \boldsymbol{0} \textit{ on } \Gamma_0\},$$
$$\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon; \boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon; \boldsymbol{v}) \sqrt{g(\varepsilon)} dx - \frac{1}{\varepsilon} \int_{\Gamma_C} p(\varepsilon)(-u_3(\varepsilon) - s(\varepsilon)) v_3 \sqrt{g(\varepsilon)} d\Gamma$$
$$= \int_\Omega f^i(\varepsilon) v_i \sqrt{g(\varepsilon)} dx + \frac{1}{\varepsilon} \int_{\Gamma_+} h^i(\varepsilon) v_i \sqrt{g(\varepsilon)} d\Gamma \; \forall \boldsymbol{v} \in V(\Omega).$$

We note that $V(\Omega)$ is a Hilbert space, with associated norm denoted by $|| \cdot ||_{1,\Omega}$. Analogously to Problem 2.2, it is straightforward that for $\varepsilon > 0$ small enough, we can apply (3), (6) and a Korn inequality (see for example[9] Th. 1.7-4) to prove the unique solvability of Problem 2.6.

3 The Koiter Contact Model for Elastic Elliptic Membranes

We now focus on the Koiter contact model for elastic elliptic membrane shells. Following[9], this is a case where S is a surface with strictly positive Gaussian curvature everywhere, and such that it is subjected to a (homogeneous) boundary condition along its entire lateral face, i.e., $\gamma_0 = \gamma$.

Let us define the space,

$$V_K(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \; \eta_i = \partial_\nu \eta_3 = 0 \text{ in } \gamma_0\},$$

and let us define the vector of unknowns $\boldsymbol{\xi}_K^\varepsilon = (\xi_{K,i}^\varepsilon)$, the trio of covariant components of the displacement field $\tilde{\boldsymbol{\xi}}^\varepsilon = \xi_{K,i}^\varepsilon \boldsymbol{a}^i$ on the points of the middle surface S. We recall the definition of the two-dimensional fourth-order elasticity tensor:

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}).$$

It can be shown that (see[9] Theorem 3.3-2) there exists $c_e > 0$ such that

$$c_e a^{\alpha\beta\sigma\tau} t_{\alpha\beta} t_{\sigma\tau} \geq \sum_{\alpha,\beta} t_{\alpha\beta}^2, \tag{10}$$

in ω, for all $t_{\alpha\beta} \in \mathbb{S}^2$. Moreover, by regularity assumption on $\boldsymbol{\theta}$, we can define

$$m_a = \max_{\alpha,\beta,\sigma,\tau \in \{1,2\}} \left\{ \|a^{\alpha\beta\sigma\tau}\|_{C(\bar{\omega})} \right\}. \tag{11}$$

On the other hand, given $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, let

$$\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,\\
\rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha(\partial_\beta \eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha \eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) + b^\tau_{\beta|\alpha}\eta_\tau,
\end{aligned}$$

denote the covariant components of the linearized change of metric tensor and linearized change of curvature tensors, respectively, both of them associated with a displacement field $\tilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i$ of the surface S. In the sequel, we assume the following scalings:

$$\lambda^\varepsilon = \lambda,\ \mu^\varepsilon = \mu,\ f^{i,\varepsilon}(\boldsymbol{x}^\varepsilon) = f^i(\boldsymbol{x}),\ h^{i,\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon h^i(\boldsymbol{x}),$$

and

$$p^\varepsilon(r^\varepsilon) = \varepsilon p(r). \tag{12}$$

Remark 3.1. Notice that this scaling leads us to consider a Lipschitz constant $L_p > 0$ *independent of* ε *for* $p(\cdot)$*. After (3) we can take* $L_p = \frac{L_p^\varepsilon}{\varepsilon}$ *and therefore we may need an additional condition of the form* $L_p^\varepsilon \leq C_L \varepsilon$*, for ensuring that the positive constant* L_p *is indeed uniform with respect to* ε*.*

Therefore, the two-dimensional Koiter's equations for elastic elliptic membrane shells in contact with a deformable obstacle are written as follows:

Problem 3.2. *Find* $\boldsymbol{\xi}^\varepsilon_K = (\xi^\varepsilon_{K,i}) \in V_K(\omega)$ *such that*

$$\varepsilon B_M(\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}) + \varepsilon^3 B_F(\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}) + \varepsilon j(\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}) = \varepsilon l(\boldsymbol{\eta})\ \forall \boldsymbol{\eta} = (\eta_i) \in V_K(\omega)$$

where, $B_M : V_K(\omega) \times V_K(\omega) \to \mathbb{R}$*,* $B_F : V_K(\omega)) \times V_K(\omega) \to \mathbb{R}$ *are the bilinear symmetric functionals*

$$\begin{aligned}
B_M(\boldsymbol{\xi}, \boldsymbol{\eta}) &:= \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy,\\
B_F(\boldsymbol{\xi}, \boldsymbol{\eta}) &:= \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\xi})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy,
\end{aligned}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in V_K(\omega)$*. Further,* $j : V_K(\omega) \times V_K(\omega) \to \mathbb{R}$ *is the non-linear (in the first argument) functional*

$$j(\boldsymbol{\xi}, \boldsymbol{\eta}) := -\int_\omega p(-\xi_3 - s)\eta_3\sqrt{a}dy,$$

and $l : V_K(\omega) \to \mathbb{R}$ *is the linear functional*

$$l(\boldsymbol{\eta}) := \int_\omega q^i \eta_i \sqrt{a}dy,$$

where

$$q^i := \int_{-1}^{1} f^i dx_3 + h^i_+, \ \textit{with } h^i_+(\boldsymbol{y}) := h^i(\boldsymbol{y},1)\ \forall \boldsymbol{y} \in \omega.$$

Notice that above, and in what follows, the M and F subscripts identify *Membrane* and *Flexural* terms, respectively. Now, let us show that Problem 3.2 has a unique solution.

Theorem 3.3. *Assume that* $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$ *and that* $\boldsymbol{f} \in [L^2(\Omega)]^3$ *and* $\boldsymbol{h} \in [L^2(\Gamma_+)]^3$*. Then, there exists* $\boldsymbol{\xi}^\varepsilon_K \in V_K(\omega)$*, unique solution of Problem 3.2.*

Proof. Let us define the non-linear operator $A : V_K(\omega) \to V_K(\omega)'$ such that

$$\langle A\boldsymbol{\xi}, \boldsymbol{\eta}\rangle = \varepsilon B_M(\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}) + \varepsilon^3 B_F(\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}) + \varepsilon j(\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}),$$

and the linear and continuous function $\boldsymbol{L} \in V_K(\omega)'$ such that

$$\langle \boldsymbol{L}, \boldsymbol{\eta}\rangle = \varepsilon l(\boldsymbol{\eta}).$$

Notice that above and in the sequel a prime $'$ indicates the topological dual of a given space, and with $\langle\, ,\, \rangle$ we indicate the dual product. Then, Problem 3.2 equivalently consists of finding $\boldsymbol{\xi}^\varepsilon_K \in V_K(\omega)$ such that

$$\langle A\boldsymbol{\xi}^\varepsilon_K, \boldsymbol{\eta}\rangle = \langle \boldsymbol{L}, \boldsymbol{\eta}\rangle\, \forall \boldsymbol{\eta} \in V_K(\omega).$$

By using the Korn inequality in[9] Th. 2.6-4, the ellipticity of $a^{\alpha\beta\sigma\tau}$ (see (10)) and properties of p derived from (3) and subsequent scalings, we find that A is a strongly monotone and Lipschitz continuous operator. As a direct consequence of the Minty-Browder Theorem, there is a unique solution of Problem 3.2.

Let us now recall the two-dimensional limit contact problem for a linearly elastic elliptic membrane shell, as $\varepsilon \to 0$ (see[28]):

Problem 3.4. *Find* $\boldsymbol{\xi} : \omega \to \mathbb{R}^3$ *such that,*

$$\begin{aligned}
&\boldsymbol{\xi} \in V_M(\omega) := H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega),\\
&\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy - \int_{\Gamma_C} p(-\xi_3 - s)\eta_3\sqrt{a}d\Gamma = \int_\omega q^i\eta_i\sqrt{a}dy\ \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega).
\end{aligned}$$

Next, we are going to give the convergence result of this section.

Theorem 3.5. *Assume the hypotheses of Theorem 3.3. Recall that* $\boldsymbol{u}^\varepsilon \in V(\Omega)$ *denotes the solution of Problem 2.2, while* $\boldsymbol{\xi}^\varepsilon_K \in V_K(\omega)$ *is the solution of Problem 3.2 and* $\boldsymbol{\xi} \in V_M(\omega)$ *represents the solution of Problem 3.4. Then it is verified that as* $\varepsilon \to 0$*, the following* *convergences hold:*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon} dx_3^\varepsilon \to \xi_\alpha \boldsymbol{a}^\alpha \text{ in } H^1(\omega), \tag{13}$$

$$\xi_{K,\alpha}^\varepsilon \boldsymbol{a}^\alpha \to \xi_\alpha \boldsymbol{a}^\alpha \text{ in } H^1(\omega), \tag{14}$$

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon} dx_3^\varepsilon \to \xi_3 \boldsymbol{a}^3 \text{ in } L^2(\omega), \tag{15}$$

$$\xi_{K,3}^\varepsilon \boldsymbol{a}^3 \to \xi_3 \boldsymbol{a}^3 \text{ in } L^2(\omega). \tag{16}$$

Proof. The proof of (13) and (15) can be found in[28] Theorem 6.2. Now, from Problem 3.2 we have

$$B_M(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\eta}) + j(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\eta}) = l(\boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} = (\eta_i) \in V_K(\omega). \tag{17}$$

Specialising $\boldsymbol{\eta} = \boldsymbol{\xi}_K^\varepsilon$, we find

$$\frac{\sqrt{a_0}}{c_e}\left(\frac{1}{c_M^2}\|\boldsymbol{\xi}_K^\varepsilon\|_{V_M(\omega)}^2 + \frac{1}{3}|\varepsilon\rho_{\alpha\beta}(\boldsymbol{\xi}_K^\varepsilon)|_{0,\omega}^2\right) + j(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}_K^\varepsilon) \le l(\boldsymbol{\xi}_K^\varepsilon) \le \|\boldsymbol{L}\|\|\boldsymbol{\xi}_K^\varepsilon\|_{V_M(\omega)}, \tag{18}$$

where a_0 is a lower bound for a (see[9] Theorem 3.3-1), c_e is the ellipticity constant in (10), c_M is the constant in a Korn inequality for the elliptic membranes case (see[9] Theorem 2.7-3) and $\|\boldsymbol{L}\| := \{\sum_i |q^i|_{0,\omega}^2\}^{\frac{1}{2}}$. We deduce that $\{\boldsymbol{\xi}_K^\varepsilon\}$ is uniformly bounded in $V_M(\omega)$ with respect to ε and, therefore, there exists a subsequence, still indexed by ε and an element $\boldsymbol{\xi}^* \in V_M(\omega)$ such that $\boldsymbol{\xi}_K^\varepsilon \rightharpoonup \boldsymbol{\xi}^*$ in $V_M(\omega)$. Going back to (18), we also find that $\{\varepsilon\rho_{\alpha\beta}(\boldsymbol{\xi}_K^\varepsilon)\}$ is uniformly bounded in $L^2(\omega)$. Therefore, there exists $\rho_{\alpha\beta}^{-1} \in L^2(\omega)$ such that (for a subsequence) $\varepsilon\rho_{\alpha\beta}(\boldsymbol{\xi}_K^\varepsilon) \rightharpoonup \rho_{\alpha\beta}^{-1}$ in $L^2(\omega)$. We will also find that $\{p(\boldsymbol{\xi}_K^\varepsilon)\}$ is uniformly bounded in $L^2(\omega)$ and there exists $\Psi \in L^2(\omega)$ such that $p(\boldsymbol{\xi}_K^\varepsilon) \rightharpoonup \Psi$ in $L^2(\omega)$. Indeed, let us first prove that there exists a constant $c > 0$ such that

$$j(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}_K^\varepsilon) \ge c||p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)||_{L^2(\omega)}^2. \tag{19}$$

In fact, for $\boldsymbol{\xi}_{K,3}^\varepsilon < -s$ we have

$$0 \le p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s) = |p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s) - p(-s)| \le L_p| - \boldsymbol{\xi}_{K,3}^\varepsilon| = L_p(-\boldsymbol{\xi}_{K,3}^\varepsilon),$$

from which

$$p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)(-\boldsymbol{\xi}_{K,3}^\varepsilon) \ge \frac{1}{L_p}[p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)]^2. \tag{20}$$

On the other hand, for $\boldsymbol{\xi}_{K,3}^\varepsilon \ge -s$, by definition, $p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s) = 0$. Then

$$p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)(-\boldsymbol{\xi}_{K,3}^\varepsilon) = [p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)]^2 = \frac{1}{L_p}[p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)]^2. \tag{21}$$

From (20) and (21) it follows that (19) holds for $c = \frac{\sqrt{a_0}}{L_p}$. Returning to (18), we deduce that

$$||p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)||_{L^2(\omega)}^2 \le \frac{L_p\|\boldsymbol{L}\|}{\sqrt{a_0}}\|\boldsymbol{\xi}_K^\varepsilon\|_{V_M(\omega)},$$

and as $\{\boldsymbol{\xi}_K^\varepsilon\}$ is uniformly bounded in $V_M(\omega)$, we conclude that

$$||p(-\boldsymbol{\xi}_{K,3}^\varepsilon - s)||_{L^2(\omega)}^2 \leq \frac{L_p\|\boldsymbol{L}\|C}{\sqrt{a_0}}.$$

Going back to (17) and passing to the limit as $\varepsilon \to 0$, we find

$$B_M(\boldsymbol{\xi}^*, \boldsymbol{\eta}) - \int_\omega \Psi\eta_3\sqrt{a}dy = l(\boldsymbol{\eta}) \ \forall \boldsymbol{\eta} \in V_K(\omega). \tag{22}$$

Notice that $V_K(\omega)$ is dense in $V_M(\omega)$. In fact, $\mathcal{D}(\omega) \subset V_K(\omega) \subset V_M(\omega)$ and $\mathcal{D}(\omega)$ is dense in $V_M(\omega)$ with its norm. Therefore, (22) is also valid for all $\boldsymbol{\eta} \in V_M(\omega)$. Further, because of the ellipticity of B_M and monotonicity of p, there exists a constant $C > 0$ such that

$$\begin{aligned} C\|\boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*\|_{V_M(\omega)}^2 &\leq B_M(\boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*, \boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*) + j(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*) - j(\boldsymbol{\xi}^*, \boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*) \\ &= B_M(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}_K^\varepsilon) - 2B_M(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}^*) + B_M(\boldsymbol{\xi}^*, \boldsymbol{\xi}^*) + j(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*) - j(\boldsymbol{\xi}^*, \boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*) \\ &= l(\boldsymbol{\xi}_K^\varepsilon) - \varepsilon^2 B_F(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}_K^\varepsilon) - 2B_M(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}^*) + B_M(\boldsymbol{\xi}^*, \boldsymbol{\xi}^*) - j(\boldsymbol{\xi}_K^\varepsilon, \boldsymbol{\xi}^*) - j(\boldsymbol{\xi}^*, \boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*), \end{aligned}$$

where in the last step we used (17) with $\boldsymbol{\eta} = \boldsymbol{\xi}_K^\varepsilon$. Then, having in mind the ellipticity of B_F and the weak convergences above, passing to the limit as $\varepsilon \to 0$ gives

$$C \lim_{\varepsilon \to 0} \|\boldsymbol{\xi}_K^\varepsilon - \boldsymbol{\xi}^*\|_{V_M(\omega)}^2 \leq l(\boldsymbol{\xi}^*) - B_M(\boldsymbol{\xi}^*, \boldsymbol{\xi}^*) - \int_\omega \Psi\xi_3^*\sqrt{a}dy = 0,$$

where in the last step we used (22) specialising $\boldsymbol{\eta} = \boldsymbol{\xi}^*$. Therefore, the convergence $\boldsymbol{\xi}_K^\varepsilon \to \boldsymbol{\xi}^*$ is strong in $V_M(\omega)$. As a consequence, we also have $p(-\xi_{K,3}^\varepsilon - s) \to p(-\xi_3^* - s)$ in $L^2(\omega)$. By uniqueness of the limit, we deduce that $\Psi = p(-\xi_3 - s)$ and therefore $\boldsymbol{\xi}^* = \boldsymbol{\xi}$, unique solution of Problem 3.4.

4 A penalized Koiter Contact problem for Elastic Elliptic Membranes

We want to approximate the solution of Problem 3.2 via a finite element method. Taking into account the high regularity in the third coordinate, and the abscense of finite elements for fourth order problems like the Argyris triangle, we will apply the intrinsic (basis free) formulation in[10] (see also[38]), based on admissible displacements of the form $\tilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i$. In terms of this displacement, we have the following expressions for the covariant components of the change of metric tensor and the change of curvature tensors:

$$\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \frac{1}{2}\left(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha\right), \quad \tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \left(\partial_{\alpha\beta}\tilde{\boldsymbol{\eta}} - \Gamma_{\alpha\beta}^\sigma \partial_\sigma \tilde{\boldsymbol{\eta}}\right) \cdot \boldsymbol{a}_3.$$

Further, if we define $\tilde{\boldsymbol{\xi}} = -\left(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3\right) \boldsymbol{a}^\alpha$, we obtain that

$$\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \frac{1}{2}\left(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \partial_\beta \boldsymbol{a}_3 + \partial_\beta \tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3 + \partial_\alpha \tilde{\boldsymbol{\xi}} \cdot \boldsymbol{a}_\beta + \partial_\beta \tilde{\boldsymbol{\xi}} \cdot \boldsymbol{a}_\alpha\right).$$

Now, let us consider the modified curvature tensor defined by

$$\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\xi}}) = \frac{1}{2}\left(\partial_\alpha\tilde{\boldsymbol{\eta}}\cdot\partial_\beta\boldsymbol{a}_3 + \partial_\beta\tilde{\boldsymbol{\eta}}\cdot\partial_\alpha\boldsymbol{a}_3 + \partial_\alpha\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{a}_\beta + \partial_\beta\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{a}_\alpha\right),\ \forall(\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{\xi}})\in\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega).$$

Notice that, above and in the sequel, the straight and boldface version of the notation of a previously defined functional space represents a triple cartesian product of the original one. For example, $\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega) := [H^1_0(\omega)]^3$. Now, Problem 3.2 can be rewritten in the following intrinsic formulation:

Problem 4.1. *Find* $(\tilde{\boldsymbol{\xi}}^\varepsilon,\tilde{\boldsymbol{\eta}}^\varepsilon)\in\mathbf{W}(\omega) := \{(\tilde{\boldsymbol{\phi}}^\varepsilon,\tilde{\boldsymbol{\varphi}}^\varepsilon)\in\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega) : \tilde{\boldsymbol{\varphi}}^\varepsilon + \left(\partial_\alpha\tilde{\boldsymbol{\phi}}^\varepsilon\cdot\boldsymbol{a}_3\right)\boldsymbol{a}^\alpha = \mathbf{0}\}$ *such that,*

$$\begin{aligned}&\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\gamma}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}^\varepsilon)\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^\varepsilon)\sqrt{a}dy + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}^\varepsilon,\tilde{\boldsymbol{\eta}}^\varepsilon)\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^\varepsilon,\tilde{\boldsymbol{\varphi}}^\varepsilon)\sqrt{a}dy\\ &\quad-\varepsilon\int_{\Gamma_C} p(-\tilde{\boldsymbol{\xi}}^\varepsilon\cdot\boldsymbol{a}_3 - s)\tilde{\boldsymbol{\phi}}^\varepsilon\cdot\boldsymbol{a}_3\sqrt{a}d\Gamma = \varepsilon\int_\omega\tilde{\boldsymbol{q}}\cdot\tilde{\boldsymbol{\phi}}^\varepsilon\sqrt{a}dy,\quad\forall(\tilde{\boldsymbol{\phi}}^\varepsilon,\tilde{\boldsymbol{\varphi}}^\varepsilon)\in\mathbf{W}(\omega),\end{aligned}$$

with $\tilde{\boldsymbol{q}} = q^i\boldsymbol{a}_i$.

We now aim to reformulate the problem from the space $\mathbf{W}(\omega)$ to the product space $\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)$. To achieve this, we introduce a penalized version of Problem 4.1 replacing the constraint $\tilde{\boldsymbol{\xi}} + (\partial_\alpha\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_3)\,\boldsymbol{a}_\alpha = \mathbf{0}$ with an additional term carrying the penalization in the variational formulation. The resulting penalized problem takes the following form.

Problem 4.2. *Find* $(\tilde{\boldsymbol{\xi}}^\varepsilon_\kappa,\tilde{\boldsymbol{\eta}}^\varepsilon_\kappa)\in\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)$ *such that,*

$$\begin{aligned}&\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\gamma}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}^\varepsilon_\kappa)\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^\varepsilon)\sqrt{a}dy + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}^\varepsilon_\kappa,\tilde{\boldsymbol{\eta}}^\varepsilon_\kappa)\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^\varepsilon,\tilde{\boldsymbol{\varphi}}^\varepsilon)\sqrt{a}dy\\ &\quad+\frac{\varepsilon}{\kappa}\int_\omega\left[\tilde{\boldsymbol{\eta}}^\varepsilon_\kappa + (\partial_\alpha\tilde{\boldsymbol{\xi}}^\varepsilon_\kappa\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right]\cdot\left[\tilde{\boldsymbol{\varphi}}^\varepsilon + (\partial_\alpha\tilde{\boldsymbol{\phi}}^\varepsilon\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right]\sqrt{a}dy\\ &\quad-\varepsilon\int_{\Gamma_C} p(-\tilde{\boldsymbol{\xi}}^\varepsilon_\kappa\cdot\boldsymbol{a}_3 - s)\tilde{\boldsymbol{\phi}}^\varepsilon\cdot\boldsymbol{a}_3\sqrt{a}d\Gamma = \varepsilon\int_\omega\tilde{\boldsymbol{q}}\cdot\tilde{\boldsymbol{\phi}}^\varepsilon\sqrt{a}dy,\quad\forall(\tilde{\boldsymbol{\phi}}^\varepsilon,\tilde{\boldsymbol{\varphi}}^\varepsilon)\in\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega),\end{aligned}$$

where $\kappa > 0$ is the penalization parameter, intended to go to zero.

We can prove existence and uniqueness of solutions for Problems 4.1 and 4.2 applying Minty-Browder Theorem. We only show the proof for Problem 4.2, since the same arguments can be used for Problem 4.1 without considering the penalized term.

Theorem 4.3. *Let* $\boldsymbol{f}\in[L^2(\Omega)]^3$ *and* $\boldsymbol{h}\in[L^2(\Gamma_+)]^3$. *Then, for each* $\kappa > 0$ *and* $\varepsilon > 0$, *Problem 4.2 admits a unique solution.*

Proof. Let us define the non-linear operator $A:\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\to(\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega))'$ such that:

$$\begin{aligned}\langle A(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle &= \varepsilon\tilde{B}_M(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\phi}}) + \varepsilon^3\tilde{B}_F((\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})) + \varepsilon\tilde{j}(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\phi}})\\ &\quad+\frac{\varepsilon}{\kappa}\int_\omega\left[\tilde{\boldsymbol{\eta}} + (\partial_\alpha\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right]\cdot\left[\tilde{\boldsymbol{\varphi}} + (\partial_\alpha\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right]\sqrt{a}dy,\end{aligned}\tag{23}$$

where

$$\begin{aligned}\tilde{B}_M(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\phi}}) &:= \int_\omega a^{\alpha\beta\sigma\tau}\tilde{\gamma}_{\sigma\tau}(\tilde{\boldsymbol{\xi}})\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\phi}})\sqrt{a}dy,\\ \tilde{B}_F((\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})) &:= \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}})\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\sqrt{a}dy,\\ \tilde{j}(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\phi}}) &:= -\int_\omega p(-\tilde{\boldsymbol{\xi}}^\varepsilon\cdot\boldsymbol{a}_3 - s)\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{a}_3\sqrt{a}dy,\end{aligned}$$

and the linear functional $\tilde{\boldsymbol{L}} \in \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)'$ such that

$$\langle\tilde{\boldsymbol{L}},\tilde{\boldsymbol{\phi}}\rangle = \varepsilon\tilde{l}(\tilde{\boldsymbol{\phi}}), \tag{24}$$

with

$$\tilde{l}(\tilde{\boldsymbol{\phi}}) = \int_\omega \tilde{\boldsymbol{q}}\cdot\tilde{\boldsymbol{\phi}}\sqrt{a}dy.$$

Then, Problem 4.2 can be formulated as follows

$$(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) \in \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega), \quad \langle A(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle = \langle\tilde{\boldsymbol{L}},\tilde{\boldsymbol{\phi}}\rangle, \quad \forall(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}) \in \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)\times\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega).$$

We note that

$$\begin{aligned}\langle A(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) - A(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}),(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) - (\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle &= \varepsilon\tilde{B}_M(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}}) + \varepsilon^3\tilde{B}_F((\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}}),(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}}))\\ &\quad + \varepsilon\int_\omega\Big[-p(-\tilde{\boldsymbol{\xi}}^\varepsilon\cdot\boldsymbol{a}_3 - s) - p(-\tilde{\boldsymbol{\phi}}^\varepsilon\cdot\boldsymbol{a}_3 - s)\Big](-\xi_3+\phi_3)\sqrt{a}dy\\ &\quad + \frac{\varepsilon}{\kappa}\int_\omega\Big[(\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}}) + (\partial_\alpha(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}})\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\Big]^2\sqrt{a}dy.\end{aligned}$$

In particular, for $\varepsilon < 1$ and κ sufficiently small, we have

$$\begin{aligned}&C_0\frac{\varepsilon^3}{3}\Big(\sum_{\alpha,\beta}\Big(\|\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}})\|^2_{L^2(\omega)} + \|\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}})\|^2_{L^2(\omega)}\Big)\\ &\qquad + \|(\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}}) + (\partial_\alpha(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}})\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\|^2_{L^2(\omega)}\Big)\\ &\le c_e^{-1}\sqrt{a_0}\Big(\varepsilon\sum_{\alpha,\beta}\Big(\|\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}})\|^2_{L^2(\omega)} + \frac{\varepsilon^3}{3}\|\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}})\|^2_{L^2(\omega)}\Big)\\ &\qquad + \frac{\varepsilon}{\kappa}\|(\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}}) + (\partial_\alpha(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}})\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\|^2_{L^2(\omega)}\Big)\\ &\le \langle A(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) - A(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}),(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) - (\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle,\end{aligned} \tag{25}$$

where $C_0 = c_e^{-1}\sqrt{a_0}$ and we use the monotonicity of p and Theorem 3.3-2 of[9].

On the other hand, by Lemma 3.3 of[39], we know that

$$\left(\sum_{\alpha,\beta}\Big(\|\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}})\|^2_{L^2(\omega)} + \|\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}})\|^2_{L^2(\omega)}\Big) + \|\tilde{\boldsymbol{\eta}} + (\partial_\alpha\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\|^2_{L^2(\omega)}\|^2_{L^2(\omega)}\right)^{\frac{1}{2}},$$

is a norm in $\mathbf{H_0^1}(\omega) \times \mathbf{H_0^1}(\omega)$ equivalent to the norm $\| \cdot \|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)}$. Then, there exists $C_1 > 0$ such that

$$\begin{aligned} C_1^2 \|(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}})\|^2_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)} \leq \sum_{\alpha,\beta} \Big(&\|\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}})\|^2_{L^2(\omega)} + \|\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}})\|^2_{L^2(\omega)} \Big) \\ &+ \|(\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}}) + (\partial_\alpha(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}) \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\|^2_{L^2(\omega)}. \end{aligned} \tag{26}$$

Hence, going back to (25), it follows that

$$\langle A(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}) - A(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}}), (\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}) - (\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}})\rangle \geq C(\varepsilon) \|(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}})\|^2_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)}, \tag{27}$$

where

$$C(\varepsilon) = \frac{C_0 C_1^2 \varepsilon^3}{3}. \tag{28}$$

Next, specialising for $\tilde{\boldsymbol{\phi}} = \mathbf{0}, \tilde{\boldsymbol{\varphi}} = \mathbf{0}$, we have that

$$\langle A(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}), (\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}})\rangle \geq C(\varepsilon) \|(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}})\|^2_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)},$$

from where

$$\frac{\langle A(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}), (\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}})\rangle}{\|((\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}))\|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)}} \to \infty, \quad \text{when} \quad \|(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}})\|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)} \to \infty.$$

Furthermore, notice that A is a Lipschitz continuous operator from $\mathbf{H_0^1}(\omega) \times \mathbf{H_0^1}(\omega)$ to $(\mathbf{H_0^1}(\omega) \times \mathbf{H_0^1}(\omega))'$ and that $A : \mathbf{H_0^1}(\omega) \times \mathbf{H_0^1}(\omega) \to (\mathbf{H_0^1}(\omega) \times \mathbf{H_0^1}(\omega))'$. Indeed, using the fact that there exists a constant upper bound, a_1, for a (see[9] p. 157),

$$\begin{aligned} &\langle A(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}) - A(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}}), (\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{w}})\rangle \\ &\leq C_2(\varepsilon, \kappa) \left\|(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}})\right\|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)} \|(\tilde{z}, \tilde{w})\|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)} + \varepsilon\sqrt{a_1}\, L_p \left\|\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}\right\|_{\mathbf{L_2}(\omega)} \|\tilde{\boldsymbol{z}}\|_{\mathbf{L_2}(\omega)} \\ &\leq C(\varepsilon, \kappa) \left\|(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}})\right\|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)} \|(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{w}})\|_{\mathbf{H_0^1}(\omega)\times\mathbf{H_0^1}(\omega)} \; \forall (\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{w}}) \in \mathbf{H_0^1}(\omega) \times \mathbf{H_0^1}(\omega), \end{aligned} \tag{29}$$

where $C_2(\varepsilon, \kappa) = \varepsilon\sqrt{a_1} \max\left\{m_a, \kappa^{-1}, \frac{\varepsilon^2}{3}\right\}$, with m_a defined in (11), and

$$C(\varepsilon, \kappa) = \varepsilon\sqrt{a_1} \max\left\{m_a, \kappa^{-1}, \frac{\varepsilon^2}{3}, L_p\right\}, \tag{30}$$

and where we have taken into account the Lipschitz continuity of p, see (3). Then, applying the Minty–Browder Theorem (see[40] Theorem 5.16), we deduce that Problem 4.2 has a unique solution. □

We now prove that the solution to Problem 4.2 converges to the solution of Problem 4.1.

Theorem 4.4. *Let $(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa}) \in \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega) \times \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)$ be the solution of Problem 4.2 and let $(\tilde{\boldsymbol{\xi}}^{\varepsilon}, \tilde{\boldsymbol{\eta}}^{\varepsilon}) \in \mathbf{W}(\omega)$ be the solution of Problem 4.1. Then*

$$\left\| \left(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa}\right) - \left(\tilde{\boldsymbol{\xi}}^{\varepsilon}, \tilde{\boldsymbol{\eta}}^{\varepsilon}\right) \right\|_{\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega) \times \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)} \to 0, \quad \textit{when} \ \ \kappa \to 0^+,$$

Proof. Consider $\tilde{\boldsymbol{\phi}}^{\varepsilon} = \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}$ and $\tilde{\boldsymbol{\varphi}}^{\varepsilon} = \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa}$ in Problem 4.2. Then,

$$\varepsilon \int_{\omega} a^{\alpha\beta\sigma\tau} \tilde{\gamma}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}) \tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}) \sqrt{a} dy + \frac{\varepsilon^3}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa}) \tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa}) \sqrt{a} dy$$

$$+ \frac{\varepsilon}{\kappa} \int_{\omega} \left[\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} + (\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha}\right]^2 \sqrt{a} dy - \varepsilon \int_{\Gamma_C} p(-\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3 - s) \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3 \sqrt{a} d\Gamma = \varepsilon \int_{\omega} \tilde{\boldsymbol{q}} \cdot \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \sqrt{a} dy.$$

Recalling (25), we deduce that

$$\begin{aligned} & C_0 \left[\sum_{\alpha,\beta} \left(\|\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa})\|^2_{L^2(\omega)} + \frac{\varepsilon^2}{3} \|\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa})\|^2_{L^2(\omega)} \right) \right. \\ & \qquad \left. + \frac{1}{\kappa} \|\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} + (\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha}\|^2_{L^2(\omega)} + \|p(-\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3 - s)\|^2_{L^2(\omega)} \right] \\ & \leq \|\tilde{\boldsymbol{L}}\| \|\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}\|_{\mathbf{H}^1(\omega)} \leq \|\tilde{\boldsymbol{L}}\| \|(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa})\|_{\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega) \times \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)}, \end{aligned} \tag{31}$$

with $\|\tilde{\boldsymbol{L}}\| := \{\sum_i |\tilde{q}_i|^2_{0,\omega}\}^{\frac{1}{2}}$. Then, we deduce that $\{\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}\}$ and $\{\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa}\}$ are uniformly bounded in $\mathbf{H}^{\mathbf{1}}(\omega)$ with respect to the κ parameter, and therefore, there exists subsequences, still indexed by κ, and $\tilde{\boldsymbol{\xi}}^{\varepsilon}_0, \tilde{\boldsymbol{\eta}}^{\varepsilon}_0 \in \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)$ such that $\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \rightharpoonup \tilde{\boldsymbol{\xi}}^{\varepsilon}_0$ and $\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} \rightharpoonup \tilde{\boldsymbol{\eta}}^{\varepsilon}_0$ in $\mathbf{H}^{\mathbf{1}}(\omega)$ as $\kappa \to 0$.

Applying the Rellich-Kondrašov theorem (see for example[41] Theorem 6.6-3), we conclude that

$$\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \to \tilde{\boldsymbol{\xi}}^{\varepsilon}_0 \text{ and } \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} \to \tilde{\boldsymbol{\eta}}^{\varepsilon}_0 \text{ in } \mathbf{L}^{\mathbf{2}}(\omega). \tag{32}$$

On the other hand, from (31) we deduce that

$$\frac{C_0}{\kappa} \|\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} + (\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha}\|^2_{L^2(\omega)} \leq \|\tilde{\boldsymbol{L}}\| \|(\tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa}, \tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa})\|_{\mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega) \times \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\omega)} \leq C(\varepsilon) \|\tilde{\boldsymbol{L}}\|^2.$$

Additionally, recall that $\| \cdot \|_{L^2(\omega)}$ is a convex and strongly lower semicontinuous functional (since it is strongly continuous). Then, by[41] Theorem 9.2-3, $\| \cdot \|_{L^2(\omega)}$ is a sequentially weakly lower semicontinuous functional. Hence, taking $\kappa \to 0$ its follows that

$$\|\tilde{\boldsymbol{\eta}}^{\varepsilon}_0 + (\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_0 \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha}\|^2_{L^2(\omega)} \leq \liminf_{\kappa \to 0} \|\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} + (\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha}\|^2_{L^2(\omega)} \leq \liminf_{\kappa \to 0} \frac{\kappa}{C_0} \|\tilde{\boldsymbol{L}}\|^2 = 0,$$

from where $\tilde{\boldsymbol{\eta}}^{\varepsilon}_0 = -(\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_0 \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha}$ a.e. in ω. Thus, $(\tilde{\boldsymbol{\xi}}^{\varepsilon}_0, \tilde{\boldsymbol{\eta}}^{\varepsilon}_0) \in \mathbf{W}(\omega)$. Moreover, we have that

$$\tilde{\boldsymbol{\eta}}^{\varepsilon}_{\kappa} + (\partial_{\alpha} \tilde{\boldsymbol{\xi}}^{\varepsilon}_{\kappa} \cdot \boldsymbol{a}_3) \boldsymbol{a}^{\alpha} \to \mathbf{0}, \quad \text{in } L^2(\omega). \tag{33}$$

Using again (31), we obtain that

$$\|p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)\|_{L^2(\omega)}^2\leq\frac{1}{C_0}\|\tilde{\boldsymbol{L}}\|^2,$$

from where there exists $\psi^{\varepsilon}\in L^2(\omega)$ and a subsequence of $\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}$, which is also denoted as $\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}$, such that

$$p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)\rightharpoonup\psi^{\varepsilon},\quad \text{in } L^2(\omega),\ \text{as } \kappa\to 0. \tag{34}$$

Additionally, we have

$$\begin{aligned}
&||p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}||_{L^2(\omega)}^2\\
&=\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}]p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)dy-\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}]\psi^{\varepsilon}dy\\
&\quad=\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}][p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)]dy\\
&\qquad+\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}]p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)dy-\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}]\psi^{\varepsilon}dy\\
&\quad\leq L_p\int_{\omega}|p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}|\,|\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}|dy\\
&\qquad+\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}]p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)dy-\int_{\omega}[p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}]\psi^{\varepsilon}dy.
\end{aligned}$$

Using (32) and (34) it follows that

$$\lim_{\kappa\to 0}||p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}||_{L^2(\omega)}^2=0, \tag{35}$$

that is $p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)\to\psi^{\varepsilon}$ in $L^2(\omega)$. Moreover, by properties of function p, we deduce that

$$\begin{aligned}
&||p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}||_{L^2(\omega)}^2\\
&\leq||p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}||_{L^2(\omega)}^2+||p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)||_{L^2(\omega)}^2\\
&\leq||p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}||_{L^2(\omega)}^2+L_p||\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}||_{\mathbf{L^2}(\omega)}^2,
\end{aligned}$$

and then, by (32) and (35), taking $\kappa\to 0$ we obtain that $||p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)-\psi^{\varepsilon}||_{L^2(\omega)}=0$, that is $p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s)=\psi^{\varepsilon}$ in $L^2(\omega)$, and we conclude that

$$p(-\tilde{\boldsymbol{\xi}}_{\kappa}^{\varepsilon}\cdot\boldsymbol{a}_3-s)\to p(-\tilde{\boldsymbol{\xi}}_{0}^{\varepsilon}\cdot\boldsymbol{a}_3-s),\quad \text{in } L^2(\omega). \tag{36}$$

Going back to Problem 4.2, considering $(\tilde{\boldsymbol{\phi}}^{\varepsilon},\tilde{\boldsymbol{\varphi}}^{\varepsilon})\in\mathbf{W}(\omega)$, taking $\kappa\to 0$ in the variational formulation for the subsequence considered in (36), using (33), and applying[41] 5.12-4 (c), it follows that

$$\begin{aligned}
&\varepsilon \int_\omega a^{\alpha\beta\sigma\tau} \tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\xi}}_0^\varepsilon) \tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^\varepsilon) \sqrt{a} dy + \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau} \tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}_0^\varepsilon, \tilde{\boldsymbol{\eta}}_0^\varepsilon) \tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^\varepsilon, \tilde{\boldsymbol{\varphi}}^\varepsilon) \sqrt{a} dy, \\
&- \varepsilon \int_{\Gamma_C} p(-\tilde{\boldsymbol{\xi}}_0^\varepsilon \cdot \boldsymbol{a}_3 - s) \tilde{\boldsymbol{\phi}}^\varepsilon \cdot \boldsymbol{a}_3 \sqrt{a} d\Gamma = \varepsilon \int_\omega \tilde{\boldsymbol{q}} \cdot \tilde{\boldsymbol{\phi}}^\varepsilon \sqrt{a} dy, \quad \forall (\tilde{\boldsymbol{\phi}}^\varepsilon, \tilde{\boldsymbol{\varphi}}^\varepsilon) \in \mathbf{W}(\omega).
\end{aligned}$$

Then $(\tilde{\boldsymbol{\xi}}_0^\varepsilon, \tilde{\boldsymbol{\eta}}_0^\varepsilon)$ is a solution of Problem 4.1. Since by Theorem 4.3 we know that the solution is unique, we conclude that $\tilde{\boldsymbol{\xi}}_0^\varepsilon = \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_0^\varepsilon = \tilde{\boldsymbol{\eta}}^\varepsilon$.

Applying again[9] Theorem 3.3-2 and[39] Lemma 3.3, it follows that

$$\begin{aligned}
&\frac{C_0 C_1^2 \varepsilon^2}{3} \|(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)\|^2_{\mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega)} \\
&\leq \tilde{B}_M(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) + \varepsilon^2 \tilde{B}_F((\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon), (\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)) \\
&\quad + \frac{1}{\kappa} \int_\omega \left[(\tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon) + (\partial_\alpha(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\right]^2 \sqrt{a} dy + \tilde{j}(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \tilde{j}(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon).
\end{aligned}$$

Finally, using that $(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon)$ is a solution of Problem 4.2, $(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}^\varepsilon)$ is a solution of Problem 4.1, and that $(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}^\varepsilon) \in \mathbf{W}(\omega)$, we obtain that

$$\begin{aligned}
&\frac{C_0 C_1^2 \varepsilon^2}{3} \|(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)\|^2_{\mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega)} \\
&\leq \tilde{B}_M(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) + \varepsilon^2 \tilde{B}_F((\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon) - (\tilde{\boldsymbol{\eta}}^\varepsilon, \tilde{\boldsymbol{\xi}}^\varepsilon), (\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon) - (\tilde{\boldsymbol{\eta}}^\varepsilon, \tilde{\boldsymbol{\xi}}^\varepsilon)) \\
&\quad + \frac{1}{\kappa} \int_\omega \left[(\tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon) + (\partial_\alpha(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\right]^2 \sqrt{a} dy + \tilde{j}(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \tilde{j}(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) \\
&= \tilde{B}_M(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) + \varepsilon^2 \tilde{B}_F((\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon), (\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)) \\
&\quad + \frac{1}{\kappa} \int_\omega \left[\tilde{\boldsymbol{\eta}}_\kappa^\varepsilon + (\partial_\alpha \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\right] \left[(\tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon) + (\partial_\alpha(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\right] \sqrt{a} dy \\
&\quad + \tilde{j}(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \tilde{j}(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \tilde{B}_M(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \varepsilon^2 \tilde{B}_F((\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}^\varepsilon), (\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)),
\end{aligned}$$

where in the last step we used that $(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}^\varepsilon) \in \mathbf{W}(\omega)$ to cancel some terms in the penalized integral. Next, specialising Problem 4.2 for $(\tilde{\boldsymbol{\phi}}^\varepsilon, \tilde{\boldsymbol{\varphi}}^\varepsilon) = (\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)$, we find

$$\begin{aligned}
&\frac{C_0 C_1^2 \varepsilon^2}{3} \|(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)\|^2_{\mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega)} \\
&\leq \tilde{l}(\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \tilde{B}_M(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) - \varepsilon^2 \tilde{B}_F((\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}^\varepsilon), (\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\eta}}_\kappa^\varepsilon - \tilde{\boldsymbol{\eta}}^\varepsilon)) \\
&\quad - \tilde{j}(\tilde{\boldsymbol{\xi}}^\varepsilon, \tilde{\boldsymbol{\xi}}_\kappa^\varepsilon - \tilde{\boldsymbol{\xi}}^\varepsilon) \to 0, \quad \text{as } \kappa \to 0^+,
\end{aligned}$$

where we last apply[41] 5.12-4 and the fact that $\tilde{\boldsymbol{\xi}}_\kappa^\varepsilon \rightharpoonup \tilde{\boldsymbol{\xi}}^\varepsilon$, $\tilde{\boldsymbol{\eta}}_\kappa^\varepsilon \rightharpoonup \tilde{\boldsymbol{\eta}}^\varepsilon$ in $\mathbf{H}_0^1(\omega)$. □

5 Numerical Analysis of the Koiter Contact Model

In this section, we present the finite element method for the penalized Problem 4.2. We begin by introducing some necessary definitions for the finite element formulation.

5.1 *The finite element method for penalized problem*

Consider a regular family of triangulations $\mathcal{T}_h$ of ω in the sense of[42] Chapter 3. The parameter $h > 0$ stands for the maximum diameter amongst the triangles in the mesh, the mesh size for short, and is intended to tend to 0. For a natural number $d \in \mathbb{N}$, we define the finite dimensional space V_h^d by

$$V_h^d = \left\{ v \in \mathcal{C}^0(\bar{\omega}) : v|_T \in \mathcal{P}_d(T)\, \forall T \in \mathcal{T}_h,\ v|_{\partial\omega} = 0 \right\},$$

comprising all the globally continuous and piecewise polynomial of degree d functions verifying the homogenous Dirichlet boundary condition, i.e. $V_h^d = \mathcal{P}_d(\mathcal{T}_h) \cap H_0^1(\omega)$. Next, we introduce the finite element space

$$\mathbf{V}_h = V_h^d \times V_h^d \times V_h^d,$$

and the following finite element formulation:

Problem 5.1. *Find* $(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h}, \tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}) \in \mathbf{V}_h \times \mathbf{V}_h \subset \mathbf{H}_\mathbf{0}^\mathbf{1}(\omega) \times \mathbf{H}_\mathbf{0}^\mathbf{1}(\omega)$, *such that*

$$\begin{aligned}
&\varepsilon \int_\omega a^{\alpha\beta\sigma\tau} \tilde{\gamma}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h}) \tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^{\varepsilon,h}) \sqrt{a} dy + \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau} \tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h}, \tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}) \tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\phi}}^{\varepsilon,h}, \tilde{\boldsymbol{\varphi}}^{\varepsilon,h}) \sqrt{a} dy \\
&\quad + \frac{\varepsilon}{\kappa} \int_\omega \left[\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h} + (\partial_\alpha \tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h} \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha \right] \cdot \left[\tilde{\boldsymbol{\varphi}}^{\varepsilon,h} + (\partial_\alpha \tilde{\boldsymbol{\phi}}^{\varepsilon,h} \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha \right] \sqrt{a} dy \\
&\quad - \varepsilon \int_{\Gamma_C} p(-\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h} \cdot \boldsymbol{a}_3 - s) \tilde{\boldsymbol{\phi}}^{\varepsilon,h} \cdot \boldsymbol{a}_3 \sqrt{a} d\Gamma = \varepsilon \int_\omega \tilde{\boldsymbol{q}} \cdot \tilde{\boldsymbol{\phi}}^{\varepsilon,h} \sqrt{a} dy, \quad \forall (\tilde{\boldsymbol{\phi}}^{\varepsilon,h}, \tilde{\boldsymbol{\varphi}}^{\varepsilon,h}) \in \mathbf{V}_h \times \mathbf{V}_h,
\end{aligned}$$

with $\tilde{\boldsymbol{q}} = q^i \boldsymbol{a}_i$.

Remark 5.2. Since our focus is in the algorithm which will follow, and for the sake of readability, we avoid the introduction of a version of the Problem 5.1 featuring projections of the various terms into the finite dimensional spaces.

We reformulate this problem by using the operator $A : \left(\mathbf{H}_\mathbf{0}^\mathbf{1}(\omega) \times \mathbf{H}_\mathbf{0}^\mathbf{1}(\omega)\right) \to \left(\mathbf{H}_\mathbf{0}^\mathbf{1}(\omega) \times \mathbf{H}_\mathbf{0}^\mathbf{1}(\omega)\right)'$ and $\tilde{\boldsymbol{L}} \in \left(\mathbf{H}_\mathbf{0}^\mathbf{1}(\omega)\right)'$ introduced in (23) and (24), respectively:

Find $(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h}, \tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}) \in \mathbf{V}_h \times \mathbf{V}_h$ such that

$$\langle A(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h}, \tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}), (\tilde{\boldsymbol{\phi}}^{\varepsilon,h}, \tilde{\boldsymbol{\varphi}}^{\varepsilon,h}) \rangle = \langle \tilde{\boldsymbol{L}}, \tilde{\boldsymbol{\phi}}^{\varepsilon,h} \rangle, \quad \forall (\tilde{\boldsymbol{\phi}}^{\varepsilon,h}, \tilde{\boldsymbol{\varphi}}^{\varepsilon,h}) \in \mathbf{V}_h \times \mathbf{V}_h.$$

Proceeding like in (27) and (29) we can show that

$$\langle A(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}) - A(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}}), (\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}) - (\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}}) \rangle \geq C(\varepsilon) \|(\tilde{\boldsymbol{\xi}} - \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\varphi}})\|^2_{\mathbf{H}_\mathbf{0}^\mathbf{1}(\omega) \times \mathbf{H}_\mathbf{0}^\mathbf{1}(\omega)}, \tag{37}$$

and

$$\left\|A(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) - A(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\right\|_{(\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega))'} \le C(\varepsilon,\kappa)\left\|(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}) - (\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\right\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)}, \tag{38}$$

for all $(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}})$, $(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}) \in \mathbf{V}_h \times \mathbf{V}_h$, where $C(\varepsilon)$ and $C(\varepsilon,\kappa)$ are defined by (28) and (30) respectively, which means that A is a strongly monotone and Lipschitz operator in $\mathbf{V}_h \times \mathbf{V}_h$, a subspace of $\mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega)$. Therefore, the Minty–Browder theorem ensures there exists a unique solution $(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}) \in \mathbf{V}_h \times \mathbf{V}_h$ to Problem 5.1.

Next, we establish an error estimate for the finite element discretization of Problem 4.2, and the convergence of the numerical solution as the mesh size h tends to zero.

Theorem 5.3. *Let $\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}\right)$ be the solution of Problem 4.2 and let $\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}\right)$ be the solution of Problem 5.1. Assume that $\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}\right) \in \mathbf{H}^2(\omega) \times \mathbf{H}^2(\omega)$. Then, there exists a constant $\tilde{C}(\varepsilon,\kappa) > 0$ such that*

$$\left\|(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}) - (\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h})\right\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)} \le \tilde{C}(\varepsilon,\kappa) h \left\|(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon})\right\|_{\mathbf{H}^2(\omega)\times\mathbf{H}^2(\omega)}. \tag{39}$$

Proof. We write Problem 4.2 as

$$\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}\right) \in \mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega),\ \langle A\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}\right), \left(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}\right)\rangle = \langle \tilde{\boldsymbol{L}},\tilde{\boldsymbol{\phi}}\rangle \quad \forall \left(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}\right) \in \mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega), \tag{40}$$

and Problem 5.1 as

$$\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}\right) \in \mathbf{V}_h \times \mathbf{V}_h,\ \langle A\left(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h}\right), \left(\tilde{\boldsymbol{\phi}}^h,\tilde{\boldsymbol{\varphi}}^h\right)\rangle = \langle \tilde{\boldsymbol{L}},\tilde{\boldsymbol{\phi}}^h\rangle \quad \forall \left(\tilde{\boldsymbol{\phi}}^h,\tilde{\boldsymbol{\varphi}}^h\right) \in \mathbf{V}_h \times \mathbf{V}_h. \tag{41}$$

By (37) and (38) we know that the operator A is strongly monotone and globally Lipschitz continuous, respectively. Further, the right hand side of (37) can be written as

$$\chi(\|(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}})\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)})\|(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}})\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)},$$

with $\chi(t) = C(\varepsilon)t$, where $C(\varepsilon)$ is defined by (28), which satisfies $\chi(0) = 0$ and $\chi(t) \to \infty$ when $t \to \infty$. Thus, we are in the framework and hypotheses of[42] Theorem 5.3.4. It follows that the solutions of (40) and (41) satisfy

$$\begin{aligned} C(\varepsilon)\left\|(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}) - (\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h})\right\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)} &= \chi\left(\left\|(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}) - (\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon,h},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon,h})\right\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)}\right) \\ &\le C(\varepsilon,\kappa) \inf_{(\tilde{\boldsymbol{\phi}}^h,\tilde{\boldsymbol{\varphi}}^h)\in\mathbf{V}_h\times\mathbf{V}_h} \left\|(\tilde{\boldsymbol{\xi}}_\kappa^{\varepsilon},\tilde{\boldsymbol{\eta}}_\kappa^{\varepsilon}) - (\tilde{\boldsymbol{\phi}}^h,\tilde{\boldsymbol{\varphi}}^h)\right\|_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)}, \end{aligned}$$

where $C(\varepsilon,\kappa)$ is defined by (30). Further, by using standard arguments of finite element interpolation (see, for example[42] Th. 3.2.1), under the assumption of additional regularity in the hypotheses of the theorem, we can establish that (39) holds with $\tilde{C}(\varepsilon,\kappa)=\frac{C(\varepsilon,\kappa)}{C(\varepsilon)}C_3$ where C_3 is a constant which depends on the domain ω, the degree d of approximation and the shape regularity constant of the family of meshes $\mathcal{T}_h$. □

Remark 5.4. Note that, for small ε *and* κ *we have* $\tilde{C}(\varepsilon,\kappa)\leq\frac{3\sqrt{a_1}c_eC_3}{\varepsilon^2\kappa\sqrt{a_0}C_1^2}$.

5.2 Numerical scheme for the nonlinear problem

In this section, we propose a fixed point iterative scheme (FPs) to approximate the finite element solution of the nonlinear Problem (5.1). To this end, we begin by splitting $A=A_0+B$ with A_0 the linear part given by

$$\langle A_0(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle=\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\gamma}_{\sigma\tau}(\tilde{\boldsymbol{\xi}})\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\phi}})\sqrt{a}dy+\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\tilde{\rho}_{\sigma\tau}(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}})\tilde{\rho}_{\alpha\beta}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\sqrt{a}dy$$
$$+\frac{\varepsilon}{\kappa}\int_\omega\left[\tilde{\boldsymbol{\eta}}+(\partial_\alpha\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right]\cdot\left[\tilde{\boldsymbol{\varphi}}+(\partial_\alpha\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right]\sqrt{a}dy,\ \forall(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\in\mathbf{V}_h\times\mathbf{V}_h,$$

and B being the nonlinear operator defined by

$$\langle B(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle=-\varepsilon\int_\omega p(-\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{a}_3-s)\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{a}_3\sqrt{a}dy,\ \forall(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}}),(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\in\mathbf{V}_h\times\mathbf{V}_h.$$

We note that, by (27), taking into account the monotonicity of p, it follows that

$$\langle A_0(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}})-A_0(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}}),(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\eta}})-(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\varphi}})\rangle\geq C(\varepsilon)\|(\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\varphi}})\|^2_{\mathbf{H}_0^1(\omega)\times\mathbf{H}_0^1(\omega)},\tag{42}$$

where $C(\varepsilon)$ is defined by (28). In what follows, for the sake of readability we shall rename the solution of Problem (5.1) as $(\boldsymbol{u}_h,\boldsymbol{r}_h)$. Further, we will focus on the specific case $p(r)=k_nr^+=k_nxH(x)$, where H denotes the Heaviside function. Recall that k_n corresponds to the Lipschitz constant of p (see (4)). Then, the operator B is defined as

$$\langle B(\boldsymbol{u},\boldsymbol{r}),(\boldsymbol{v},\boldsymbol{t})\rangle=-k_n\varepsilon\int_\omega(-\boldsymbol{u}\cdot\boldsymbol{a}_3-s)H(-\boldsymbol{u}\cdot\boldsymbol{a}_3-s)\boldsymbol{v}\cdot\boldsymbol{a}_3\sqrt{a}dy\,\forall(\boldsymbol{u},\boldsymbol{r}),\ (\boldsymbol{v},\boldsymbol{t})\in\mathbf{V}_h\times\mathbf{V}_h.$$

Next, we proceed to rewriting the nonlinear problem as follows:

Find $(\boldsymbol{u}_h,\boldsymbol{r}_h)\in\mathbf{V}_h\times\mathbf{V}_h$ such that

$$\langle A_0(\boldsymbol{u}_h,\boldsymbol{r}_h),(\boldsymbol{v},\boldsymbol{t})\rangle+\langle B(\boldsymbol{u}_h,\boldsymbol{r}_h),(\boldsymbol{v},\boldsymbol{t})\rangle=\langle\tilde{\boldsymbol{L}},\boldsymbol{v}\rangle\quad\forall\,(\boldsymbol{v},\boldsymbol{t})\in\mathbf{V}_h\times\mathbf{V}_h,\tag{43}$$

We now describe the corresponding iterative scheme to approximate the solution of problem (43):

Given an initial guess $(\boldsymbol{u}^0, \boldsymbol{r}^0) \in \mathbf{V}_h \times \mathbf{V}_h$, *we iteratively compute* $(\boldsymbol{u}^k, \boldsymbol{r}^k) \in \mathbf{V}_h \times \mathbf{V}_h$, $k \geq 1$, *until convergence.*
Thus, for each step $k \geq 0$, *the pair* $(\boldsymbol{u}^k, \boldsymbol{r}^k) \in \mathbf{V}_h \times \mathbf{V}_h$ *is known, and serves as data for computing the next iteration* $(\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1}) \in \mathbf{V}_h \times \mathbf{V}_h$, *as the solution of the linear problem*

$$\begin{aligned}\langle A_0(\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1}), (\boldsymbol{v}, \boldsymbol{t})\rangle - k_n\varepsilon \int_\omega \left((-\boldsymbol{u}^{k+1}) \cdot \boldsymbol{a}_3 - s\right) H(-\boldsymbol{u}^k \cdot \boldsymbol{a}_3 - s) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx \\ = -\varepsilon \int_\omega \tilde{\boldsymbol{q}} \cdot \boldsymbol{v} \sqrt{a}\, dx, \qquad \forall (\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h. \quad (44)\end{aligned}$$

The left hand side of (44) defines a bilinear form on $[\mathbf{V}_h \times \mathbf{V}_h]^2$ for the variables $(\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1})$ and $(\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h$, which is continuous and coercive. The right hand side is a continuous linear form for the variable $(\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h$. Then, by the Lax-Milgram Theorem, there exists a unique $(\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1}) \in \mathbf{V}_h \times \mathbf{V}_h$ solution of (44).

To shorten the notation in what follows, let us now introduce $z^\ell = -\boldsymbol{u}^\ell \cdot \boldsymbol{a}_3 - s$ for $\ell \in \mathbb{N}$. Then $(\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1})$ satisfies

$$\langle A_0\left(\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1}\right), (\boldsymbol{v}, \boldsymbol{t})\rangle - k_n\varepsilon \int_\omega z^{k+1} H(z^k) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx = -\varepsilon \int_\omega \tilde{\boldsymbol{q}} \cdot \boldsymbol{v} \sqrt{a}\, dx,$$

for all $(\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h$. On the other hand, introducing $z_h = -\boldsymbol{u}_h \cdot \boldsymbol{a}_3 - s$, the finite element solution $(\boldsymbol{u}_h, \boldsymbol{r}_h) \in \mathbf{V}_h \times \mathbf{V}_h$ satisfies

$$\langle A_0\left((\boldsymbol{u}_h, \boldsymbol{r}_h)\right), (\boldsymbol{v}, \boldsymbol{t})\rangle - k_n\varepsilon \int_\omega z_h H(z_h) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx = -\varepsilon \int_\omega \tilde{\boldsymbol{q}} \cdot \boldsymbol{v} \sqrt{a}\, dx,$$

for all $(\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h$. Substracting the previous two equations we obtain

$$\begin{aligned}&\langle A_0\left((\boldsymbol{u}_h, \boldsymbol{r}_h) - (\boldsymbol{u}^{k+1}, \boldsymbol{r}^{k+1})\right), (\boldsymbol{v}, \boldsymbol{t})\rangle = k_n\varepsilon \int_\omega \left(z_h H(z_h) - z^{k+1} H(z^k)\right) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx \\ &= N_1(\boldsymbol{v}) + N_2(\boldsymbol{v}) + N_3(\boldsymbol{v}), \qquad (45)\end{aligned}$$

where we have defined

$$\begin{aligned}N_1(\boldsymbol{v}) &:= k_n\varepsilon \int_\omega \left(z_h H(z_h) - z^k H(z^k)\right) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx \\ N_2(\boldsymbol{v}) &:= k_n\varepsilon \int_\omega \left(z^k - z_h\right) H(z^k) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx \\ N_3(\boldsymbol{v}) &:= k_n\varepsilon \int_\omega \left(z_h - z^{k+1}\right) H(z^k) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx\end{aligned}$$

for all $(\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h$. Using the inequality $|xH(x) - yH(y)| \leq |x - y|$, the bound $0 \leq H(x) \leq 1$, and the upper bound a_1 for a, we can derive the following estimates:

$$|N_1(\boldsymbol{v})| = \left| k_n\varepsilon \int_\omega \left(z_h H(z_h) - z^k H(z^k)\right) \boldsymbol{v} \cdot \boldsymbol{a}_3 \sqrt{a}\, dx \right| \leq k_n\varepsilon \sqrt{a_1} \|z_h - z^k\|_{L^2(\omega)} \|\boldsymbol{v} \cdot \boldsymbol{a}_3\|_{L^2(\omega)}.$$

$$|N_2(\boldsymbol{v})| = \left| k_n\varepsilon \int_\omega \left(z^k - z_h\right) H(z^k)\boldsymbol{v}\cdot\boldsymbol{a}_3\sqrt{a}\,dx \right| \leq k_n\varepsilon\sqrt{a_1}\|z_h - z^k\|_{L^2(\omega)}\|\boldsymbol{v}\cdot\boldsymbol{a}_3\|_{L^2(\omega)}.$$

Hence

$$\begin{aligned}|N_1(\boldsymbol{v})| + |N_2(\boldsymbol{v})| &\leq 2k_n\varepsilon\sqrt{a_1}\|z_h - z^k\|_{L^2(\omega)}\|\boldsymbol{v}\cdot\boldsymbol{a}_3\|_{L^2(\omega)}\\ &\leq 2k_n\varepsilon\sqrt{a_1}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{L^2(\omega)}\|\boldsymbol{v}\|_{L^2(\omega)},\end{aligned}$$

where in the last step we used the definition of z_h and z^k and that $\|\boldsymbol{a}_3\| = 1$. Now, let us choose $\boldsymbol{v} = \boldsymbol{u}_h - \boldsymbol{u}^{k+1}$. Then, it follows that:

$$|N_1(\boldsymbol{u}_h - \boldsymbol{u}^{k+1})| + |N_2(\boldsymbol{u}_h - \boldsymbol{u}^{k+1})| \leq 2k_n\varepsilon\sqrt{a_1}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{L^2(\omega)}\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{L^2(\omega)}, \tag{46}$$

and

$$\begin{aligned}N_3(\boldsymbol{u}_h - \boldsymbol{u}^{k+1}) &= k_n\varepsilon\int_\omega \left(z_h - z^{k+1}\right) H(z^k)(\boldsymbol{u}_h - \boldsymbol{u}^{k+1})\cdot\boldsymbol{a}_3\sqrt{a}\,dx\\ &= -k_n\varepsilon\int_\omega \left[\left(\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\right)\cdot\boldsymbol{a}_3\right]^2 H(z^k)\sqrt{a}\,dx \leq 0.\end{aligned} \tag{47}$$

Using the coercitivity (42), and estimates (46), (47) in (45), we obtain

$$\begin{aligned}&C(\varepsilon)(\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|^2_{\mathbf{H}^1(\omega)} + \|\boldsymbol{r}_h - \boldsymbol{r}^{k+1}\|^2_{\mathbf{H}^1(\omega)}) \leq \|(u_h - \boldsymbol{u}^{k+1}, r_h - \boldsymbol{r}^{k+1})\|^2_{\mathbf{H}^1(\omega)\times\mathbf{H}^1(\omega)}\\ &\quad\leq \langle A_0\left((\boldsymbol{u}_h,\boldsymbol{r}_h) - (\boldsymbol{u}^{k+1},\boldsymbol{r}^{k+1})\right), (\boldsymbol{u}_h - \boldsymbol{u}^{k+1}, \boldsymbol{r}_h - \boldsymbol{r}^{k+1})\rangle\\ &\quad= N_1(\boldsymbol{u}_h - \boldsymbol{u}^{k+1}) + N_2(\boldsymbol{u}_h - \boldsymbol{u}^{k+1}) + N_3(\boldsymbol{u}_h - \boldsymbol{u}^{k+1})\\ &\quad\leq 2k_n\varepsilon\sqrt{a_1}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{\mathbf{H}^1(\omega)}\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{\mathbf{H}^1(\omega)}.\end{aligned}$$

Therefore,

$$\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|^2_{\mathbf{H}^1(\omega)} + \|(\boldsymbol{r}_h - \boldsymbol{r}^{k+1})\|^2_{\mathbf{H}^1(\omega)} \leq \frac{2k_n\varepsilon\sqrt{a_1}}{C(\varepsilon)}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{\mathbf{H}^1(\omega)}\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{\mathbf{H}^1(\omega)},$$

and we conclude that

$$\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{\mathbf{H}^1(\omega)} \leq \frac{2k_n\varepsilon\sqrt{a_1}}{C(\varepsilon)}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{\mathbf{H}^1(\omega)},$$

and

$$\|(\boldsymbol{r}_h - \boldsymbol{r}^{k+1})\|^2_{\mathbf{H}^1(\omega)} \leq \frac{2k_n\varepsilon\sqrt{a_1}}{C(\varepsilon)}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{\mathbf{H}^1(\omega)}\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{\mathbf{H}^1(\omega)}.$$

Then, by (28), we deduce that

$$\|(\boldsymbol{r}_h - \boldsymbol{r}^{k+1})\|^2_{\mathbf{H}^1(\omega)} \leq \frac{6k_n\sqrt{a_1}c_e}{C_1^2\sqrt{a_0}\varepsilon^2}\|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{\mathbf{H}^1(\omega)}\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{\mathbf{H}^1(\omega)},$$

where C_1 is defined in (26), which proves the following result.

Theorem 5.5. *If*

$$0 < \lambda := \frac{6k_n\sqrt{a_1}c_e}{C_1^2\sqrt{a_0}\varepsilon^2} < 1$$

then the fixed point scheme (44) produces a sequence $\{(\boldsymbol{u}^k, \boldsymbol{r}^k)\}$ that converges in $\mathbf{H}_0^1(\omega) \times \mathbf{H}_0^1(\omega)$ to the solution $(\boldsymbol{u}_h, \boldsymbol{r}_h)$ of Problem (43), for any initial guess $(\boldsymbol{u}_0, \boldsymbol{r}_0)$. Additionally, $\boldsymbol{u}^k \to \boldsymbol{u}_h$ linearly in $\mathbf{H}_0^1(\omega)$ and we have

$$\|\boldsymbol{u}_h - \boldsymbol{u}^{k+1}\|_{\mathbf{H^1}(\omega)} \leq \lambda \|\boldsymbol{u}_h - \boldsymbol{u}^k\|_{\mathbf{H^1}(\omega)}.$$

Remark 5.6. Alternatively, the same iterative scheme can be recovered by applying Newton's method to the following minimization problem:

$$(\boldsymbol{u}_h, \boldsymbol{r}_h) = \operatorname{argmin}_{(\boldsymbol{u},\boldsymbol{r})\in\mathbf{V}_h\times\mathbf{V}_h} J(\boldsymbol{u}, \boldsymbol{r}), \tag{48}$$

where the functional $J : \mathbf{V}_h \times \mathbf{V}_h \to \mathbb{R}$ is defined by

$$J(\boldsymbol{u}, \boldsymbol{r}) = \frac{1}{2}\langle A_0(\boldsymbol{u}, \boldsymbol{r}), (\boldsymbol{u}, \boldsymbol{r})\rangle - k_n \frac{\varepsilon}{2} \int_\omega (-\boldsymbol{u}\cdot\boldsymbol{a}_3 - s)^2\, H(-\boldsymbol{u}\cdot\boldsymbol{a}_3 - s)\sqrt{a}dy - \langle \tilde{\boldsymbol{L}}, \boldsymbol{u}\rangle.$$

In fact, it can be shown that:

$$D_{(\boldsymbol{v},\boldsymbol{t})} J(\boldsymbol{u}, \boldsymbol{r}) = \langle A_0(\boldsymbol{u}, \boldsymbol{r}), (\boldsymbol{v}, \boldsymbol{t})\rangle + \langle B(\boldsymbol{u}, \boldsymbol{r}), (\boldsymbol{v}, \boldsymbol{t})\rangle - \langle \tilde{\boldsymbol{L}}, \boldsymbol{v}\rangle.$$

Then, to approximate a solution of (48), we apply Newton's method to the equation $D_{(\boldsymbol{v},\boldsymbol{t})} J(\boldsymbol{u}, \boldsymbol{r}) = 0$. This leads to the following iterative scheme:

Given an initial guess $(\boldsymbol{u}^0, \boldsymbol{r}^0) \in \mathbf{V}_h \times \mathbf{V}_h$, solve

$$D_{((\boldsymbol{u}^{k+1},\boldsymbol{r}^{k+1})-(\boldsymbol{u}^k,\boldsymbol{r}^k))}(D_{(\boldsymbol{v},\boldsymbol{t})} J(\boldsymbol{u}^k, \boldsymbol{r}^k)) = -D_{(\boldsymbol{v},\boldsymbol{t})} J(\boldsymbol{u}^k, \boldsymbol{r}^k)\, \forall (\boldsymbol{v}, \boldsymbol{t}) \in \mathbf{V}_h \times \mathbf{V}_h, \tag{49}$$

for $k \geq 0$ until one of the usual convergence criteria is met.

It can be verified that (49) is equivalent to the fixed point scheme (44). Moreover, it can be shown that the mapping $D_{(\boldsymbol{du},\boldsymbol{dr})} DJ_{(\boldsymbol{v},\boldsymbol{t})}$ is Lipschitz continuous with respect to the pair $(\boldsymbol{du}, \boldsymbol{dr}) \in \mathbf{V}_h \times \mathbf{V}_h$. Therefore, by applying[43] *Theorem 6.3, Chapter IV, we obtain quadratic convergence of Newton's method (49), provided that the initial guess is sufficiently close to the solution.*

6 Numerical Simulations

In this section we present numerical simulations in the study of the Problem 5.1, which will show the performance of our code and experimentally confirm the convergence theoretical results obtained in previous sections. Our implementation benefits from the FreeFEM++ programming environment. Once the results of the simulations are obtained, the 2D plots are represented with

Matlab* and the 3D renders are built with the help of the free software Paraview†. Our code is available at https://github.com/ContactShells/KoiterShellContact.

We have selected a set of four experiments to demonstrate the behavior of the model and, in particular, to verify the convergence results in Theorem 3.5 and Theorem 5.3, regarding the convergences in terms of ε and h, respectively. Some data is common to all simulations, though. To begin with, ω will always be the unit disk centered in $(0,0)$ and S is the spherical cap of radius $r=1$ in a sphere of radius $R=2$, given by $\boldsymbol{\theta}(x_1,x_2)=\sqrt{4-x_1^2-x_2^2}$, for $(x_1,x_2)\in\bar{\omega}$. We can see the mesh of the reference configuration of S in Figure 1. We consider a Koiter elliptic membrane shell and obstacle setting in which the forces act in compression and the initial gap is $s=0.25$. Besides, we have taken the Lamé coefficients

$$\lambda=12000,\ \ \mu=4000.$$

It should be noted that here and in what follows, for simplicity, we do not indicate the units associated with the data and unknowns. Further, in all the examples we will take constant forces, which ensures that the hypothesis on data are verified.

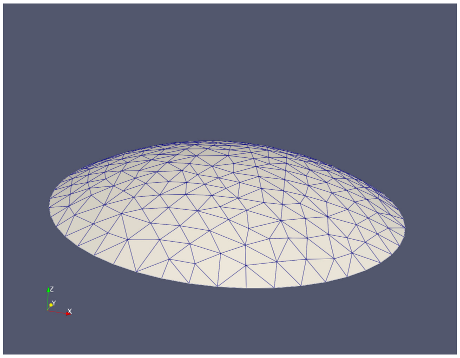

Figure 1. Mesh of the reference configuration of the middle surface S.

Experiment 1: *We observe the variation of the equilibrium states of the deformed middle surface of a Koiter elliptic shell as the applied forces increase their strength. Specifically, we take* $-1.28\times10^5\leq q_3\leq -1\times10^3$. *For this experiment we use a fixed thickness parameter* $\varepsilon=0.1$ *and a fixed rigidity of the obstacle* $k_n=400000$. *In Figure 2 it is represented the flexion of a diameter of the shell and in Figure 3 we can see the rendered deformed middle surface at the weakest, middle and strongest values of the force, in the varying interval‡. We can see that for its weakest strength, the force is not sufficient to overcome the concave curvature of S, while for the middle strength the deformed shell gets in contact with the obstacle (not shown) and take its*

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[*] https://www.mathworks.com/

[†] https://www.paraview.org/

[‡] A movie clip showing the deformed configurations for the other intermediate values of the applied forces can be consulted in[44].

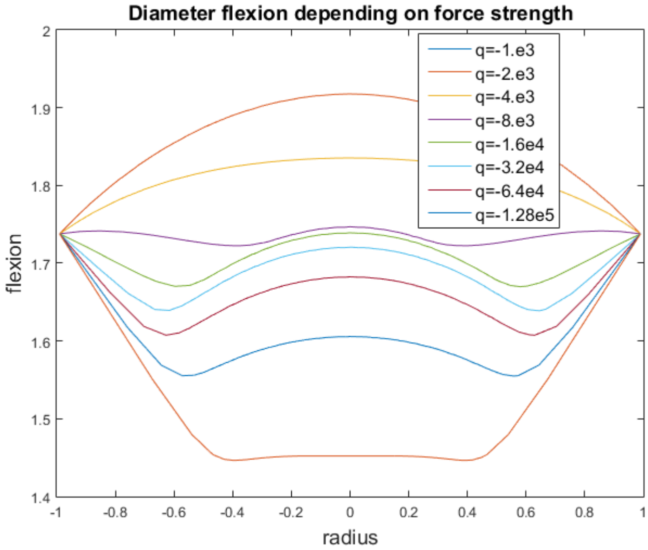

Figure 2. Flexion of a diameter of the Koiter shell with varying applied forces.

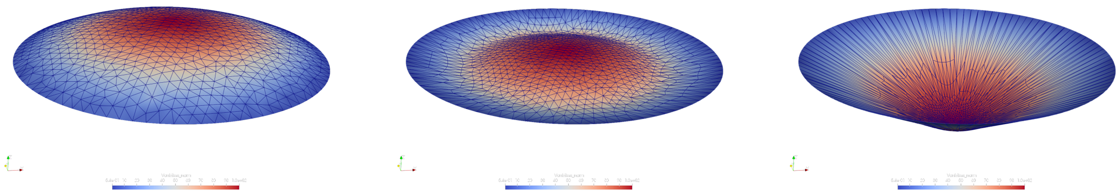

Figure 3. Von Mises norm on the deformed configuration of the Koiter elliptic shell. From left to right, increasing strength of the applied forces.

form. For the strongest force strength, the deformed configuration is like if the obstacle was not even there.

Experiment 2: *We observe the variation of the equilibrium states of the deformed middle surface of the Koiter elliptic shell as the rigidity of the obstacle declines. Specifically, we take* $6.9 \times 10^4 \leq \kappa_n \leq 1.07 \times 10^6$. *For this experiment we use a fixed thickness parameter* $\varepsilon = 0.1$ *and a fixed force strength* $q_3 = -50000$. *In Figure 4 it is represented the flexion of a diameter of the shell and in Figure 5 we can see the rendered deformed middle surface at the highest, middle and weakest values of the rigidity, in the varying interval*[§]. *We can see that for the strongest*

[§] A movie clip showing the deformed configurations for the other intermediate values of the rigidity can be consulted at [44].

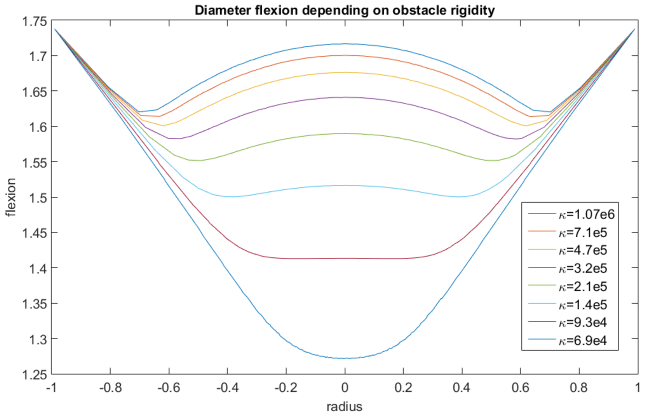

Figure 4. Flexion of a diameter of the Koiter shell with varying rigidity of the obstacle.

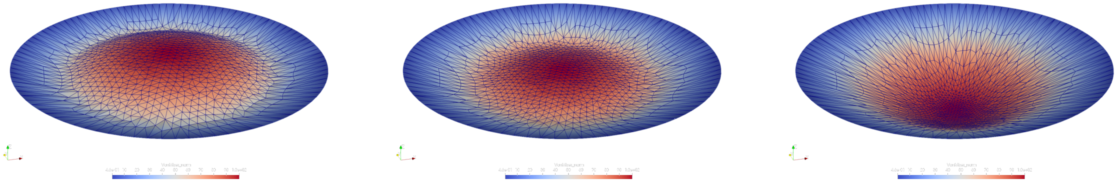

Figure 5. Von Mises norm on the deformed configuration of the Koiter elliptic shell. From left to right, decreasing rigidity of the obstacle.

rigidity, the shell adapts to the shape of the obstacle but cannot penetrate, while for the middle value the deformed shell has penetrated the obstacle, but still showing adaptation to its shape. For the weakest rigidity, the deformed configuration is like if the obstacle was not even there.

EXPERIMENT 3: *We observe the variation of the equilibrium states of the deformed middle surface of the Koiter elliptic shell as the thickness tends to zero. Specifically, we take* $0.6 \times 10^{-2} \leq \varepsilon \leq 3.2$*. For this experiment we use a fixed rigidity of the obstacle* $k_n = 400000$ *and a fixed force strength* $q_3 = -50000$*. In Figure 6(left) it is represented with discontinuous lines the flexion of a diameter of the shell and in continuous line the flexion of the same diameter of a pure elliptic membrane shell. We observe the pointwise convergence as* $\varepsilon \to 0$*. Moreover, in Figure 6(right) we observe the convergence in the norm of the space* $V_M(\omega)$*, which is an experimental verification of Theorem 3.5. In Figure 7 we can see the rendered deformed middle surface of the Koiter*

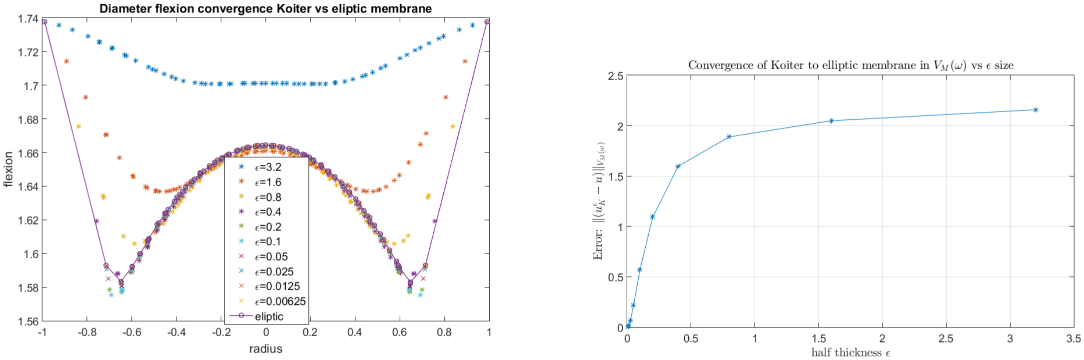

Figure 6. Left: Flexion of a diameter of the Koiter elliptic shell with varying thickness (discontinuous lines) and elliptic membrane shell (continuous line). Right: Asymptotic convergence in the norm of $V_M(\omega)$ as $\varepsilon \to 0$.

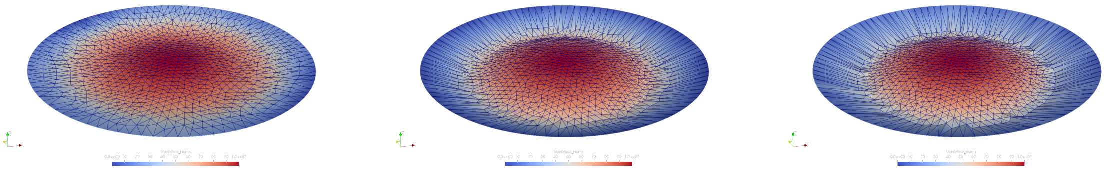

Figure 7. Von Mises norm on the deformed configuration of the Koiter shell (left and middle) and elliptic membrane (right).

elliptic shell at the largest and middle values of ε*, and the deformed middle surface of the elliptic membrane shell*[¶]*.*

EXPERIMENT 4: *We measure the numerical approximation error as* $h \to 0$*. To do this, we consider as exact solution one with a very refined mesh and compute the approximation error in the* H^1 *norm as* $h \to 0$*. We observe a linear rate of convergence in agreement with the conclusions of Theorem 5.3*[‖]*. For this experiment we used fixed values for thickness parameter,* $\varepsilon = 0.1$*, rigidity of the obstacle,* $k_n = 400000$*, and force strength,* $q_3 = -50000$*.*

[¶] A movie clip showing the deformed configurations for the other intermediate values of the thickness can be viewed at[44].

[‖] A movie clip showing the deformed configurations from using a very coarse mesh up to the refined mesh considered as exact solution can be viewed at[44].

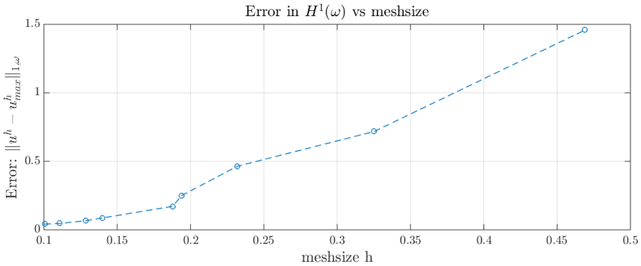

Figure 8. Convergence in the approximation error as $h \to 0$.

7 Conclusions and Outlook

We have found and mathematically justified a two-dimensional Koiter model for elastic elliptic membrane shells in normal compliance contact with a deformable foundation. The justification is based on a convergence result which shows that solution to this model converges to the solution of the same two-dimensional limit model as three-dimensional solution of the contact problem for an elastic elliptic membrane shell, when the thickness tends to zero. We have also provided a numerical scheme with finite elements, based on penalization and basis-free intrinsic formulations and found a priori error estimates. We have implemented this method into FreeFem++ and discussed numerical simulations which provide empirical evidence of the theoretical results. This work represents the first step of the program which will continue with the case of generalized membranes and flexural shells under the same contact conditions. Once completed, it will represent a major advantage from the practical applications point of view since, for normal compliance contact, one could always use a Koiter model regardless of the geometry and the boundary conditions, knowing that the solution will be a good approximation of the actual situation. Therefore, future work will be devoted to the study of the remaining cases of Koiter elastic shells in normal compliance contact, which are the generalized membranes and the flexural shells. On a longer term we also intend to study the unilateral frictionless case and further, we shall include viscoelasticity, dynamic problems and frictional conditions, in some cases related to tribological effect such as wear and adhesion.

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