A finite element discretization of fractional problems using graded meshes

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Abstract

In this paper, we deal with the finite element approximation of a homogeneous Dirichlet boundary value problem for fractional powers of symmetric second–order elliptic operators on a two–dimensional domain Ω. We employ the diagonalization technique introduced in [Banjai, Melenk, Nochetto, Otárola, Salgado, Schwab, Foundations of Computational Mathematics (2019) 19: 901–962], which proposes a semi–discretization in the extended variable of a truncated Caffarelli–Silvestre extension. This approach decouples the problem into the solution of independent second–order reaction–diffusion equations in Ω, several of which may become singularly perturbed. For the case where $\Omega = (0,1)^2$, we propose to approximate all the decoupled problems by bilinear finite elements over a unique layer adapted, suitably graded, rectangular mesh, which can be designed independently of the eventual singular perturbation parameters. We prove the convergence of the proposed scheme and show numerical examples confirming the theoretical results.

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1. Introduction

We are interested in finite element approximations of the Dirichlet problem for fractional powers of a symmetric second–order elliptic operator. Given a domain Ω in $\mathbb{R}^2$, a real $s \in (0,1)$, and a function f, the model problem reads as follows: given $f \in \mathbb{H}^{-s}(\Omega)$, find $u \in \mathbb{H}^s(\Omega)$ solution of

$$\begin{aligned} \mathcal{L}^s u = f \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with the spaces $\mathbb{H}^s$ and its dual $\mathbb{H}^{-s}$ to be introduced later, see Section 3.1.

We consider, for simplicity, the operator $\mathcal{L}$ of the form

$$\mathcal{L}v = -\Delta v + \bar{c}(x)v, \tag{2}$$

with $\bar{c}(x) \in L^\infty(\Omega)$ a non-negative function defined on Ω. More generally, operators with a diffusion coefficient could also be considered.

The main difficulty in obtaining efficient numerical methods for (1) is that $\mathcal{L}^s$ is a non–local operator [1, 2]. One of the most studied non–local operators is the *fractional Laplacian* $(-\Delta)^s$, subject to $s \in (0,1)$, with homogeneous Dirichlet boundary conditions, due to its physical applications involving long-range or anomalous diffusion. For example, it is used in modeling the flow of certain particles in porous media (see [3]). Caffarelli and Silvestre, in [2], localize problem (1), for $\mathcal{L} = -\triangle$, by means of a non–uniformly elliptic PDE posed in one more spatial dimension. They showed that any power $s \in (0,1)$ of the fractional Laplacian in $\mathbb{R}^d$ can be realized as the Dirichlet–to–Neumann map of an extension to the upper half–space $\mathbb{R}^{d+1}_+$, which, in what follows, we call the *local extended problem*. This result was expanded by Cabré and Tan [1] and by Stinga and Torrea [4] to consider bounded domains Ω and more general operators, thus obtaining an extended problem posed on the semi-infinite cylinder $\mathcal{C} := \Omega \times (0,\infty)$.

Nochetto, Otárola and Salgado in [5], proposed to approximate the solution $u(x)$ of (1) by taking the trace $\mathcal{U}(\cdot,0)$ on $\Omega \times \{0\}$ of an approximation to the solution $\mathcal{U}$ of the local extended problem (see subsection 3.2 for details). Indeed, they analyzed the extended problem in the framework of weighted Sobolev spaces, and motivated by the rapid decay of $\mathcal{U}$, they considered a truncation $\mathcal{C}_\mathcal{Y} = \Omega \times [0,\mathcal{Y}]$ of $\mathcal{C}$ and approximated $\mathcal{U}$ there by discretizing with first order tensor product finite elements. Subsequently, in [6], these authors, together with Banjai, Melenk, and Schwab, extended the previous results in several directions. In particular, they proposed a novel diagonalization technique that decouples the degrees of freedom introduced by a Galerkin (semi–)discretization in the extended variable. This technique reduces the y–semidiscrete Caffarelli-Silvestre extension to the solution of independent second-order reaction-diffusion equations posed on Ω, some of which are singularly perturbed. By introducing an hp finite element approximation of these reaction-diffusion problems, this decoupling allowed them to establish exponential convergence for analytic data f without assuming boundary compatibility (for a comprehensive discussion, see [6, Section 9]).

This paper is mainly motivated by the observation that singularly perturbed reaction–diffusion problems on a square can be almost optimally approximated in the energy norm by an h version of piecewise bilinear finite elements using appropriate meshes (graded meshes) designed independently of the perturbation parameter [7, 8]. Then, we start with the diagonalization technique from [6], and propose a strategy to design a unique graded mesh on $\Omega = (0,1)^2$ to approximate the sequence of reaction–diffusion problems coming from the semi–discretization of the extended problem in the truncated cylinder $\mathcal{C}_\mathcal{Y}$. All these numerical solutions are then combined as in [6] to obtain an approximation of the solution of (1). Our assumptions on the right–hand side f are that it is $\mathcal{C}^2(\overline{\Omega})$ and it satisfies the compatibility condition (55), that is, that f vanishes on the vertices of the square.

Our error estimates are linear up to a logarithmic factor in the number of reaction–diffusion equations to be solved. However, it is important to study the approximation error in terms of the number of degrees of freedom. In particular, it follows from Theorem 6.1 that to obtain an error of almost order $O(M^{-1})$, we need to solve M reaction-diffusion problems, each of them having $O(M^{\frac{3}{2}})$ degrees of freedom. Then, with a total number of $O(M^{\frac{5}{2}})$ degrees of freedom, we get an accuracy of $O(M^{-1})$ (up to logarithmic factors). This is slightly better than the complexity of the h version of the finite element method for a regular three-dimensional problem. We notice that the discretization of the M reaction–diffusion problems leads to M linear systems with matrices of the form $\mu_i A_1 + A_0$ and on the right-hand side $\zeta_i b$, with fixed matrices A_0 and A_1, and a fixed vector b. The coefficients μ_i and ζ_i are computed at the beginning of the process. Therefore, we believe that this approach can be combined with suitable parallelization algorithms to obtain better performance, but we do not delve into this issue in depth in this work.

It is well known that the standard finite element method on uniform or quasi–uniform meshes produces poor results for the approximation of singularly perturbed reaction–diffusion problems. In this paper, we will consider the use of graded meshes that were introduced in [7], which are a kind of adapted mesh designed with a priori knowledge of the exact solution. In Section 2.2 we briefly review other meshes introduced in the literature.

To obtain almost linear convergence in h to the solution of (1), our approach requires superlinear approximations of each one of the reaction–diffusion problems. In view of that, in Proposition 5.1, we show that a local post-processing of the bilinear finite element solution on graded meshes yields a superconvergent approximation that is almost uniform with respect to the singular perturbation parameter. We obtain that result as a consequence of a supercloseness property proved in [8] when the grading parameter defining the graded meshes is large enough. The technique for obtaining robust superconvergence results has previously been used in [9], in the case of singularly perturbed convection–diffusion problems. Similar results for singularly perturbed convection–diffusion or reaction–diffusion problems on Shishkin meshes were obtained in [10, 11, 12].

The remainder of the paper is structured as follows. Section 2 introduces the notation, followed by a brief discussion on fractional problems and the finite element approximation of singularly perturbed reaction–diffusion equations. In Section 3, we introduce the model problem and its discretization which is based on the Caffarelli–Silvestre extension. In addition, some auxiliary results are presented. Section 4 includes estimates of the semi–discretization error. In Section 5 we deal with the finite element approximation of singularly perturbed reaction–diffusion equations on graded meshes. In particular we show how a higher order approximation can be obtained from the computed solution by a simple local post–processing. Our main result on the error estimate for the proposed approximation of (1) is presented in Section 6. Finally, Section 7 contains numerical experiments that confirm the theoretical results of Sections 5 and 6.

2. Notation and preliminaries

Throughout the paper, we adopt the following notation. For a domain D, we use standard notation for L^p and Sobolev spaces, as well as their respective norms and seminorms, namely,

$$\|u\|_{L^p(D)} := \left(\int_D |u|^p\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$
$$\|u\|_{L^\infty(D)} := \inf\{C > 0 : |u(x)| \leq C \; a.e.\},$$

$$\|u\|_{m,D} := \left\{ \sum_{|\alpha|\le m} \|\mathcal{D}^\alpha u\|^2_{L^2(D)} \right\}^{1/2}, \quad |u|_{m,D} := \left\{ \sum_{|\alpha|=m} \|\mathcal{D}^\alpha u\|^2_{L^2(D)} \right\}^{1/2}.$$

In particular, $\|u\|_{0,D}$ denotes the L^2-norm of u over D. When $D = \Omega$, and no confusion can arise, we will write $\|u\|_0$ instead of $\|u\|_{0,\Omega}$.

For a rectangle R, $\mathcal{P}_k(R)$ and $\mathcal{Q}_k(R)$ denote the spaces of polynomials of total degree less than or equal to k and polynomials of degree less than or equal to k in each variable, respectively, over R.

In addition, C will denote a constant that may depend on the fractional power s or the discretization parameters σ and η introduced in Subsection 3.4, and which is independent of the mesh sizes and singular parameters in reaction-diffusion problems. The value of C might change with each occurrence. The notation $a \lesssim b$ means $a \le Cb$ and $a \sim b$ signifies $a \lesssim b \lesssim a$.

2.1. Background of Fractional Laplacian

If the space under consideration is the whole space $\mathbb{R}^n$, the fractional Laplacian operator $(-\Delta)^s$ is defined for a function in $\mathcal{S}$ and $s \in (0,1)$ as [13]

$$\begin{aligned}(-\Delta^s)u(x) &= C(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \\ &= C(n,s) \lim_{\varepsilon \searrow 0} \int_{\mathcal{C}B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy\end{aligned} \tag{3}$$

where $P.V.$ means "in the principle value sense" as defined by the latter equation and $C(n,s)$ is the constant

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\zeta_1}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}.$$

An equivalent definition can be given in terms of Fourier transform, see [13, Section 3]: for a function $u \in \mathcal{S}$ we have

$$(-\Delta^s)u(x) = \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}u\right) \tag{4}$$

where $\mathcal{F}$ denotes the Fourier transform.

However, there is no unique way to define $(-\Delta)^s w$ for functions w defined on a bounded domain Ω. One possibility is to suitably extend w to a function u in the whole space $\mathbb{R}^n$ and to equivalently use the definitions (3) or (4).

In relation to this, [14] considers the integral formulation (3) and restrict it to functions w supported on Ω. The resulting operator is called the *integral* fractional Laplacian and is denoted by $(-\Delta)_I^s w$.

Another possibility is to consider the so–called *regional* fractional Laplacian $(-\Delta)_R^s w$ that is defined by restricting the Riesz integral (3) to Ω. This operator is known to be the infinitesimal generator of the so–called censored stable Lévy processes [15, 16].

In this paper we use a third approach, which leads to the *spectral* fractional Laplacian and corresponds to the fractional powers of the Dirichlet Laplace operator in the sense of the spectral theory. Caffarelli–Silvestre's result [2] was proved for this operator allowing a localization via a local problem posed on the semi–infinity cylinder $\Omega \times (0, +\infty)$. Following [6], we exploit this localization to obtain a numerical approximation of the spectral fractional Laplacian using suitably graded meshes on Ω.

2.2. Numerical approximation of singularly perturbed reaction-diffusion problems

A key ingredient in our approach is the robust finite element approximation of singularly perturbed reaction–diffusion equations on $\Omega = (0,1)^2$ coming from the diagonalization process applied to the local extended problem. It is well known that the standard finite element methods for singularly perturbed problems produce very poor results when uniform or quasi–uniform meshes are used unless they are sufficiently refined. Consequently, these kinds of meshes are not useful in practical applications, and therefore several alternatives of appropriately adapted meshes have been considered in many papers. Very well known are the Shishkin's type meshes (see [10, 11]) consisting of piecewise uniform meshes with a similar number of elements close to and away from the layers. Transition points are suitably introduced to separate fine and coarse zones in the mesh. Shishkin meshes were also used to approximate interior layers in convection–diffusion–reaction equations with non–smooth coefficients, see, for example [17]. Other very used meshes are Bakhvalov meshes [10] which are generated by equidistributing a monitor function related to the exponential boundary layer. This kind of mesh has been recently used in [18] for the approximation of an integral boundary value problem of non–linear singularly perturbed differential equations where it is reported to produce better error estimates than those obtained for Shishkin meshes. Furthermore, a curvature–based monitor function is used in [19] to effectively approximate a stationary convection–reaction–diffusion equation, and the authors have successfully applied a r–refinement procedure for adapting the mesh in time in the non–stationary case. We also mention [20] where the equidistribution of an error monitor function was used to generate adaptive grids in time to obtain a uniformly convergent scheme for a parabolic system of reaction–diffusion equations.

In this paper, we consider graded meshes as introduced in [7] for the finite element approximation of singularly perturbed reaction–diffusion equations with homogeneous Dirichlet boundary conditions. There, it has been proved that to obtain an expected bound for the error in the energy norm, these meshes can be defined independently of the singular perturbation parameter ε ([7, Corollary 4.5]). We exploit this property here, since it allows us to approximate, using a single appropriately defined finite element mesh in space, all the decoupled reaction–diffusion problems that appear when the diagonalization technique of [6] is applied to the approximation of problem (1).

3. The model problem

In this Section we firstly introduce the fractional powers $\mathcal{L}^s$ and the Caffarelli–Silvestre extension, and secondly we describe in detail our proposed discretization, which is based on the diagonalization technique introduced in [6].

3.1. Fractional Powers of Elliptic Operators

The power $\mathcal{L}^s$, as in [6, 5], is defined following the spectral theory. Consider the countable collection of eigenpairs $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times H_0^1(\Omega)$, of the problem

$$a_\Omega(\varphi, v) = \lambda(\varphi, v)_{L^2(\Omega)} \;\forall v \in H_0^1(\Omega) \tag{5}$$

where $a_\Omega(\cdot,\cdot)$ is the inner product on $H_0^1(\Omega)$ induced by $\mathcal{L}$ given by

$$a_\Omega(w, v) = \int_\Omega (\nabla w \cdot \nabla v + cwv)\, dx' \tag{6}$$

with real eigenvalues λ_k enumerated in increasing order, counting multiplicities. It is assumed that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $(H_0^1(\Omega), a_\Omega(\cdot,\cdot))$. Then, for $s \geq 0$, we introduce the spaces

$$\mathbb{H}^s(\Omega) := \left\{ w = \sum_{k=1}^{\infty} w_k\varphi_k :\; \|w\|^2_{\mathbb{H}^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s w_k^2 < \infty \right\}$$

while $\mathbb{H}^{-s}(\Omega)$ denotes the dual space of $\mathbb{H}^s(\Omega)$.

It is known that for functions $w = \sum_k w_k\varphi_k \in \mathbb{H}^1(\Omega)$, the operator $\mathcal{L} : \mathbb{H}^1(\Omega) \to \mathbb{H}^{-1}(\Omega)$ takes the form $\mathcal{L}w = \sum_k \lambda_k w_k \varphi_k$. Then, for $s \in (0,1)$ and $w = \sum_k w_k\varphi_k \in \mathbb{H}^s(\Omega)$, the operator $\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$ is naturally defined by

$$\mathcal{L}^s w = \sum_{k=1}^{\infty} \lambda_k^s w_k \varphi_k.$$

3.2. The local extended problem

To achieve an effective computational discretization scheme, following [6], we consider a strategy proposed by Caffarelli and Silvestre [2], and subsequently extended by Cabré and Tan [1] and Stinga and Torrea [4] for bounded domains Ω, to localize it. This strategy involves solving the following singular elliptic boundary value problem posed on the extended cylinder $\mathcal{C} = \Omega \times (0, +\infty)$:

$$\begin{aligned} -\text{div}\,(y^\alpha \nabla \mathcal{U}) + \bar{c}(x) y^\alpha \mathcal{U} &= 0 \qquad \text{in } \mathcal{C} \\ \mathcal{U} &= 0 \qquad \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha}\mathcal{U} &= d_s f \qquad \text{on } \Omega \times \{0\} \end{aligned} \tag{7}$$

where $\partial_L\mathcal{C} := \partial\Omega \times (0,\infty)$ is the lateral boundary of $\mathcal{C}$, $d_s := 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)} > 0$ is a normalization constant and $\alpha := 1-2s \in (-1,1)$. The conormal exterior derivative of $\mathcal{U}$ at $\Omega \times \{0\}$ is defined by

$$\partial_{\nu^\alpha}\mathcal{U} = -\lim_{y\to 0+} y^\alpha \partial_y \mathcal{U}. \tag{8}$$

The limit in (8) is understood in the distributional sense [1, 2].

In order to analyze the problem (7) we need to introduce additional spaces. Throughout the text, we denote $x = (x', y) \in \mathcal{C}$ with $x' \in \Omega$ and $y > 0$. If $D \subset \mathbb{R}^n$, let $x = (x', y) \in D$ where y denotes the last variable in $\mathbb{R}^n$, we define $L^2(y^\alpha, D)$ as the Lebesgue space for the measure $|y|^\alpha\, dx$ and the weighted Sobolev space

$$H^1(y^\alpha, D) = \left\{ w \in L^2(y^\alpha, D) : \, |\nabla w| \in L^2(y^\alpha, D) \right\}$$

where ∇w is the gradient in weak sense of w (since $x = (x', y)$ here, we have $\nabla = \left(\nabla_{x'}, \frac{\partial}{\partial y}\right)$). We equip $H^1(y^\alpha, D)$ with the norm

$$||w||_{H^1(y^\alpha,D)} = \left(||w||^2_{L^2(y^\alpha,D)} + ||\nabla w||^2_{L^2(y^\alpha,D)} \right)^{\frac{1}{2}}. \tag{9}$$

Define the weighted Sobolev space

$$\overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}) : \, w = 0 \text{ on } \partial_L \mathcal{C} \right\} \tag{10}$$

and the bilinear form $a_\mathcal{C} : \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}) \times \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}) \to \mathbb{R}$ by

$$a_\mathcal{C}(v, w) = \int_\mathcal{C} y^\alpha (\nabla v \cdot \nabla w + cvw) \, dx' \, dy. \tag{11}$$

It can be proven as a consequence of a Poincaré's inequality that $a_\mathcal{C}(\cdot, \cdot)$ is continuous and coercive. Consequently, it induces an inner product on $\overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$ and the energy norm $\| \cdot \|_\mathcal{C}$:

$$\|v\|_\mathcal{C}^2 := a_\mathcal{C}(v, v) \sim \|\nabla v\|^2_{L^2(y^\alpha, \mathcal{C})}. \tag{12}$$

The weak formulation of (7) reads as follows: find $\mathcal{U} \in \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$ such that

$$a_\mathcal{C}(\mathcal{U}, \mathcal{V}) = d_s \langle f, tr\, \mathcal{V} \rangle \quad \forall\, \mathcal{V} \in \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}), \tag{13}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $L^2(\Omega)$ and $\operatorname{tr} \mathcal{V}$ is the trace $\mathcal{V}|_{\Omega \times \{0\}}$.

The connection between both problems follows from this fundamental result (see [1, Proposition 2.2] and [4, Theorem 1.1]): given $f \in \mathbb{H}^{-s}(\Omega)$, let $u \in \mathbb{H}^s(\Omega)$ be the solution of (1). If $\mathcal{U} \in \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$ solves (7) then

$$u = \operatorname{tr} \mathcal{U} \qquad \text{and} \qquad d_s \mathcal{L}^s u = \partial_{\nu^\alpha} \mathcal{U} \qquad \text{in } \Omega.$$

3.3. Semi–discretization of the extended problem

Let $\mathcal{Y} > 0$ and $M \in \mathbb{N}$. Given a partition $\mathcal{G}^M$ of $[0, \mathcal{Y}]$ into M subintervals, we will define in this subsection a semidiscrete approximation $\mathcal{U}_M$ of the solution $\mathcal{U}$ of (13), with $\mathcal{U}_M$ supported on the truncated cylinder $\mathcal{C}_\mathcal{Y} := \Omega \times (0, \mathcal{Y})$.

Let $S^1_{\{\mathcal{Y}\}} \left((0, \mathcal{Y}), \mathcal{G}^M \right)$ be the space of piecewise linear functions on $\mathcal{G}^M$ that vanish at $y = \mathcal{Y}$. We consider the space

$$\mathbb{V}_M = H^1_0(\Omega) \otimes S^1_{\{\mathcal{Y}\}} \left((0, \mathcal{Y}), \mathcal{G}^M \right).$$

Functions in $\mathbb{V}_M$ can be extended by 0 to the entire cylinder $\mathcal{C}$ and thus we can consider $\mathbb{V}_M$ as a subspace of $\overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$, that is, $\mathbb{V}_M \subset \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$. We can obtain the approximation $\mathcal{U}_M$ as the solution of the semidiscrete problem: find $\mathcal{U}_M \in \mathbb{V}_M$ such that

$$a_{\mathcal{C}}(\mathcal{U}_M, \mathcal{V}) = d_s \langle f, \operatorname{tr} \mathcal{V} \rangle \qquad \forall \mathcal{V} \in \mathbb{V}_M. \tag{14}$$

Let $\{(\mu_i, v_i)\}_{i=1}^M \subset \mathbb{R} \times S^1_{\{\mathcal{Y}\}}\left((0,\mathcal{Y}), \mathcal{G}^M\right) \setminus \{0\}$ be the set of eigenpairs defined by

$$\mu_i \int_0^{\mathcal{Y}} y^\alpha v_i'(y) w'(y)\, dy = \int_0^{\mathcal{Y}} y^\alpha v_i(y) w(y)\, dy \qquad \forall w \in S^1_{\{\mathcal{Y}\}}\left((0,\mathcal{Y}), \mathcal{G}^M\right)$$

with $\{v_i\}$ normalized such that

$$\int_0^{\mathcal{Y}} y^\alpha v_i'(y) v_j'(y)\, dy = \delta_{ij}, \qquad \int_0^{\mathcal{Y}} y^\alpha v_i(y) v_j(y)\, dy = \mu_i \delta_{ij}.$$

Then it can be easily verified that we can write

$$\mathcal{U}_M(x', y) = \sum_{i=1}^M U_i(x') v_i(y) \tag{15}$$

with $U_i \in H_0^1(\Omega)$, $i = 1, \ldots, M$ being the solutions of the problems

$$\mu_i \left(\nabla U_i, \nabla V\right) + ((1 + \bar{c}(x))U_i, V) = d_s v_i(0) \langle f, V \rangle \qquad \forall V \in H_0^1(\Omega). \tag{16}$$

Problems (16) are of reaction–diffusion type, and they become singularly perturbed when their diffusion coefficients, the eigenvalues μ_i, are small. The magnitude of these eigenvalues depends on the partition $\mathcal{G}^M$. In particular, for the partition introduced in the next section we have the following upper and lower bounds: for $i = 1, \ldots, M$ it holds

$$\mu_i \leq \mathcal{Y}^2 (1 - \alpha^2)^{-1}, \tag{17}$$

and

$$\mu_i \gtrsim (\log M)^{-\alpha\sigma}\, l_{min}^2. \tag{18}$$

Here, l_{min} denotes the minimal length of the intervals in $\mathcal{G}^M$, and the constant involved depends only on s. Estimates (17) follow from [6, Lemma 18] while we refer to Remark 3.3 later on for a proof of (18). Thus, if l_{min} is small, we see that a number of the reaction–diffusion problems (16) may become singularly perturbed.

3.4. The full discretization

In this subsection, we introduce our finite element approximation based on the semi–discretization previously introduced. Now we restrict ourselves to the case $\Omega = (0,1)^2$. Let σ be a parameter satisfying

$$1 - s < \sigma < 1. \tag{19}$$

Let $M \in \mathbb{N}$ and $\mathcal{Y} = c \log M$, with the constant c to be chosen later. We then define the partition $\mathcal{G}^M = \{I_i\}_{i=1}^M$ in $[0, \mathcal{Y}]$, where $I_i = [y_{i-1}, y_i]$ with

$$y_i = \mathcal{Y} \left(\frac{i}{M} \right)^{\frac{1}{1-\sigma}}, \quad i = 0, \dots, M. \tag{20}$$

A discrete approximation $\mathcal{U}_{M,N}$ of $\mathcal{U}_M$ defined in (15) is constructed by means of finite element discretizations of the problems (16). Let $N \in \mathbb{N}$. We introduce a graded mesh $\mathcal{T}_N$ of Ω obtained as a tensor product of partitions of the interval $[0,1]$ into $2N$ subintervals. Given a parameter η satisfying

$$\frac{3}{4} \leq \eta < 1, \tag{21}$$

let $\xi_0, \xi_1, \dots, \xi_N$ be the grid points on the interval $[0, \frac{1}{2}]$ given by

$$\xi_i = \frac{1}{2} \left(\frac{i}{N} \right)^{\frac{1}{1-\eta}}, \quad i = 0, \dots, N. \tag{22}$$

This partition is extended to a grid $\{\xi_0, \xi_1, \dots, \xi_N, \dots, \xi_{2N}\}$ of $[0,1]$ by setting $\xi_i = 1 - \xi_{2N-i}$ for $i = N+1, \dots, 2N$. For $1 \leq i, j \leq 2N$, let $R_{ij} = [\xi_{i-1}, \xi_i] \times [\xi_{j-1}, \xi_j]$. Then we obtain a graded mesh $\mathcal{T}_N = \{R_{ij}\}_{i,j=1}^{2N}$ of $\overline{\Omega}$. Additionally, we set $h_i = \xi_i - \xi_{i-1}$.

Remark 3.1. *The intervals $I_i, i = 2, \dots, M$, of the partition $\mathcal{G}^M$, satisfy*

$$|I_i| \leq C \mathcal{Y} \frac{1}{M} y^\sigma = C (\log M) \frac{1}{M} y^\sigma, \qquad \forall y \in I_i, \tag{23}$$

with a constant C depending only on s.

Indeed, with $i \geq 2$, for some $\zeta \in (i-1, i)$ we have

$$\begin{aligned} y_i - y_{i-1} &= \left[\left(\frac{i}{M} \right)^{\frac{1}{1-\sigma}} - \left(\frac{i-1}{M} \right)^{\frac{1}{1-\sigma}} \right] \mathcal{Y} \\ &= \frac{1}{1-\sigma} \left(\frac{\zeta}{M} \right)^{\frac{\sigma}{1-\sigma}} \frac{1}{M} \mathcal{Y}. \end{aligned}$$

But

$$\left(\frac{\zeta}{M}\right)^{\frac{\sigma}{1-\sigma}} = \left(\frac{i-1}{M}\right)^{\sigma} \left(\frac{\zeta}{i-1}\right)^{\frac{\sigma}{1-\sigma}} \left(\frac{M}{i-1}\right)^{-\frac{\sigma^2}{1-\sigma}}.$$

Then, (23) *follows from* $\mathcal{Y} = c\log M$*,*

$$\left(\frac{\zeta}{i-1}\right)^{\frac{\sigma}{1-\sigma}} \leq 2^{\frac{\sigma}{1-\sigma}}, \qquad \left(\frac{M}{i-1}\right)^{-\frac{\sigma^2}{1-\sigma}} \leq 1$$

and

$$\left(\frac{i-1}{M}\right)^{\sigma} \leq y^{\sigma} \quad \forall y \in I_i.$$

Remark 3.2. *For* $i \geq 2$ *we also have*

$$y \leq C\left(\log M\right)^{\sigma} z \qquad \forall y, z \in I_i. \tag{24}$$

Indeed, we have from (23) *that*

$$y_i \leq y_{i-1} + C\left(\log M\right)\frac{1}{M}y_{i-1}^{\sigma}.$$

Since $y_{i-1} \geq y_1 = \mathcal{Y}\left(\frac{1}{M}\right)^{\frac{1}{1-\sigma}}$*, and then* $y_{i-1}^{\sigma-1} \leq c(logM)^{\sigma-1}M$*, we obtain*

$$y_i \leq y_{i-1}\left(1 + C\left(\log M\right)^{\sigma}\right)$$

which implies (24)*.*

Associated with $\mathcal{T}_N$, we introduce the standard piecewise bilinear finite element space

$$V_N = \{v \in H_0^1(\Omega) : v|_{R_{ij}} \in \mathcal{Q}_1(R_{ij}), 1 \leq i,j \leq 2N\}, \tag{25}$$

where $\mathcal{Q}_1(R)$ denotes the space of bilinear functions on the rectangle R. Now we can define $U_{i,N}$, $i = 1, \ldots, M$, as the solutions of problems: find $U_{i,N} \in V_N$ such that

$$\mu_i\left(\nabla U_{i,N}, \nabla V\right) + ((1+\bar{c}(x))U_{i,N}, V) = d_s v_i(0)\langle f, V\rangle \qquad \forall V \in V_N. \tag{26}$$

For each $U_{i,N}$ we will define in Section 5 a post–processed $U_{i,N}^*$ with improved approximation properties. Then, similar to (15) we define

$$\mathcal{U}_{M,N}(x', y) = \sum_{i=1}^{M} U_{i,N}^*(x')v_i(y), \tag{27}$$

and finally, the approximation of u is given by

$$u_{M,N}(x') = \text{tr } (\mathcal{U}_{M,N}(x', y)) = \sum_{i=1}^{M} U^*_{i,N}(x') v_i(0). \tag{28}$$

Remark 3.3. *With the definitions introduced in this subsection, we can prove* (18)*. Using a standard rescaling argument $y = |I_1|\hat{y}$ to map intervals $[0,1]$ onto I_1, and the equivalence of norms for linear functions on I_1, we have for the eigenfunctions v_i (defined in Subsection 3.3)*

$$\|v_i'\|_{L^2(y^\alpha, I_1)} \sim |I_1|^{-1} \|v_i\|_{L^2(y^\alpha, I_1)}.$$

Using another scaling argument and the equivalence (24) *on interval I_i for $i \geq 2$, we have*

$$\|v_i'\|_{L^2(y^\alpha, I_i)} \sim (\log M)^{\frac{\alpha\sigma}{2}} |I_i|^{-1} \|v_i\|_{L^2(y^\alpha, I_i)}.$$

Therefore, by squaring and adding the previous inequalities, we obtain

$$\mu_i = \|v_i\|^2_{L^2(y^\alpha, (0, \mathcal{Y}))} \geq C (\log M)^{-\alpha\sigma} (\min |I_i|)^2 \|v_i'\|^2_{L^2(y^\alpha, I_i)}$$

and taking into account that $\|v_i'\|_{L^2(y^\alpha, I_i)} = 1$ and $\min |I_i| = |I_1|$, we have

$$\mu_i = \|v_i\|^2_{L^2(y^\alpha, (0, \mathcal{Y}))} \geq C (\log M)^{-\alpha\sigma} |I_1|^2$$

which proves (18)*.*

Remark 3.4. *The factor $v_i(0)$ which appears on the right-hand side of the problem* (16) *and its discretization* (26)*, can be bounded by following [6, Lemma 17]. Taking into account that $v(\mathcal{Y}) = 0$ and $\|v_i'\|_{L^2(y^\alpha, (0, \mathcal{Y}))} = 1$ we have*

$$|v_i(0)| = \left| \int_0^{\mathcal{Y}} v'(y)\, dy \right| = \left| \int_0^{\mathcal{Y}} y^{-\frac{\alpha}{2}} y^{\frac{\alpha}{2}} v'(y)\, dy \right| \leq \frac{\mathcal{Y}^{\frac{1-\alpha}{2}}}{(1-\alpha)^{\frac{1}{2}}} \leq C (\log M)^{\frac{1-\alpha}{2}}$$

with the constant C depending on s and independent of M.

Remark 3.5. *Similarly to Remarks 3.1 and 3.2, for all* $R_{ij} \in \mathcal{T}_N$, *we can prove that*

$$h_i \leq C\frac{1}{N}x_1^\eta, \qquad h_j \leq C\frac{1}{N}x_2^\eta, \qquad \forall (x_1, x_2) \in R_{ij} \tag{29}$$

and

$$x_1 \leq Cw_1, \qquad x_2 \leq Cw_2, \qquad \forall (x_1, x_2), (w_1, w_2) \in R_{ij}. \tag{30}$$

On the other hand it is easy to check that

$$\frac{h_i}{h_{i+1}} \leq C, \quad \forall i = 0, \dots, 2N-1. \tag{31}$$

Here, C *is a constant depending only on* η.

3.5. Some preliminaries for the error analysis

The error estimate starts with the trace inequality (see [6, Subsection 2.2])

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C_{tr}\|\mathcal{U} - \mathcal{U}_{M,N}\|_{\mathcal{C}},$$

and then, by the triangle inequality, we have

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C_{tr}\left(\|\mathcal{U} - \mathcal{U}_M\|_{\mathcal{C}} + \|\mathcal{U}_M - \mathcal{U}_{M,N}\|_{\mathcal{C}}\right). \tag{32}$$

In Section 4 we will obtain the estimate

$$\|\mathcal{U} - \mathcal{U}_M\|_{\mathcal{C}} \leq C\left(\log M\right)^m \frac{1}{M}\|f\|_0$$

with an exponent m to be defined later.

On the other hand, since

$$\mathcal{U}_M(x', y) - \mathcal{U}_{M,N}(x', y) = \sum_{i=1}^{M}\left(U_i(x', y) - U_{i,N}^*(x', y)\right)v_i(y)$$

we have, following [6, eq. (6.5)], that

$$\|\mathcal{U}_M - \mathcal{U}_{M,N}\|_{\mathcal{C}}^2 = \sum_{i=1}^{M}\|U_i - U_{i,N}^*\|_{\mu_i,\Omega}^2, \tag{33}$$

where the norm $\|\cdot\|_{\mu_i,\Omega}$ is the energy norm associated with problem (26):

$$\|v\|^2_{\mu_i,\Omega} = \|v\|^2_{\mu_i} := \mu_i \|\nabla v\|^2_0 + \|(1+\bar{c}(x))^{\frac{1}{2}} v\|^2_0.$$

We observe that, in order to obtain a linear order of convergence for our discretization, we require superlinear estimates for the approximations of the (singularly perturbed) reaction–diffusion problems defining U_i. In Section 5, we will see that this can be achieved if the parameter η, defining the graded meshes, satisfies (21).

4. Error estimate in the extended domain

In this section, we estimate the semi–discretization error $\|\mathcal{U} - \mathcal{U}_M\|_{\mathcal{C}}$. In order to do that, we need to define an interpolation operator for functions in $C^2([0,\mathcal{Y}], L^2(\Omega))$.

Given a Sobolev space X, following [6], we consider a piecewise linear interpolation operator $\pi^1_{y,\{\mathcal{Y}\}}$ defined over a grid $\mathcal{G}^M$ on $[0,\mathcal{Y}]$ for functions $v \in \mathcal{C}^2([0,\mathcal{Y}], X)$. On the interval I_1, $\pi^1_{y,\{\mathcal{Y}\}} v$ is defined by interpolating v at points $y_1/2$ and y_1; on intervals I_i with $1 < i < M$, the interpolation is at points y_{i-1} and y_i; and finally, on I_M it interpolates at y_{M-1} and is enforced to vanish at y_M, $\left(\pi^1_{y,\{\mathcal{Y}\}} v\right)(y_M) = 0$.

Let

$$\omega_{\theta,\gamma}(y) = y^\theta e^{\gamma y}.$$

We will write y^α and $\omega_{\alpha,0}(y)$ interchangeably. For a function $v \in L^2(I,X)$, where X is a Hilbert space and I is a real interval, we introduce the notation

$$\|v\|_{L^2(\omega_{\theta,\gamma},I;X)} = \left(\int_I \omega_{\theta,\gamma}(y)\|v(y)\|^2_X \, dy\right)^{\frac{1}{2}}.$$

When there is no confusion, we will omit the space X writing

$$\|v\|_{L^2(\omega_{\theta,\gamma},I)} = \|v\|_{L^2(\omega_{\theta,\gamma},I;X)}.$$

For a function $\mathcal{V}(x',y)$, with $x' \in \Omega$ and $y \in I$, such that for each y it holds $\mathcal{V}(\cdot,y) \in X$, we write

$$\|\mathcal{V}\|_{L^2(\omega_{\theta,\gamma},\Omega\times I)} := \left(\int_I \omega_{\theta,\gamma}(y)\|\mathcal{V}(\cdot,y)\|^2_X \, dy\right)^{\frac{1}{2}}.$$

We need interpolation error estimates for $\pi^1_{y,\{\mathcal{Y}\}}$. More precisely, we have the classical local estimates

$$\begin{aligned}\left\|v-\pi^1_{y,\{\mathcal{Y}\}}v\right\|_{L^2(\omega_{0,0},I_i)} &\leq C|I_i|\,\|v'\|_{L^2(\omega_{0,0},I_i)}\\ \left\|(v-\pi^1_{y,\{\mathcal{Y}\}}v)'\right\|_{L^2(\omega_{0,0},I_i)} &\leq C|I_i|\,\|v''\|_{L^2(\omega_{0,0},I_i)}\end{aligned}\tag{34}$$

for each interval I_i, for functions $v \in H^1(I,X)$ and $v \in H^2(I,X)$, respectively. On the interval I_1, we will use the weighted error estimates

$$\begin{aligned}\left\|v-\pi^1_{y,\{\mathcal{Y}\}}v\right\|_{L^2(\omega_{\alpha,0},I_1)} &\leq C|I_1|^{\beta}\,\|v'\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)}\\ \left\|(v-\pi^1_{y,\{\mathcal{Y}\}}v)'\right\|_{L^2(\omega_{\alpha,0},I_1)} &\leq C|I_i|^{\beta}\,\|v''\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)}\end{aligned}\tag{35}$$

which are proven in [6, eqs. (A.6) and (A.4)].

In view of [6, eq. (6.10)], $\mathcal{U}$ can be seen as a function in $C^2([0,\mathcal{Y}],L^2(\Omega))\cap C^2([0,\mathcal{Y}],H^1_0(\Omega))$ and thus it makes sense to consider $\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}$.

In the proof of the next result, we will use the following estimates for $\mathcal{U}$ taken from [6, Theorem 1]. Let γ be a fixed positive parameter satisfying $\gamma < 2\sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the problem (5). We have

$$\begin{aligned}\|\partial_y\mathcal{U}\|_{L^2(\omega_{\alpha-2\tilde{\nu},\gamma},\mathcal{C})} &\leq C\|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)}\\ \|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\tilde{\nu},\gamma},\mathcal{C})} &\leq C\|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)}\\ \|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\nu,\gamma},\mathcal{C})} &\leq C\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}\end{aligned}\tag{36}$$

with $0 \leq \tilde{\nu} < s = (1-\alpha)/2$ and $0 \leq \nu < 1+s$.

Proposition 4.1. *Assume $f \in L^2(\Omega)$. We consider the approximation $\mathcal{U}_M$ defined by (14), supported on the truncated cylinder $\mathcal{C}_{\mathcal{Y}}$, obtained using the grid $\mathcal{G}^M$ of $[0,\mathcal{Y}]$ defined by (20), where $\mathcal{Y} = c\log M$ with $c > \frac{3}{\gamma}$. Then it follows that*

$$\|\mathcal{U}-\mathcal{U}_M\|_{\mathcal{C}} \leq C(\log M)^{\frac{3+\alpha\sigma}{2}}\frac{1}{M}\|f\|_0,\tag{37}$$

where C is a constant depending only on s.

Proof. From the Galerkin orthogonality, we have

$$\|\mathcal{U}-\mathcal{U}_M\|_{\mathcal{C}} \leq \left\|\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right\|_{\mathcal{C}}.\tag{38}$$

It follows from the Poincaré's inequality [6, ineq. (2.7)] for functions in $\overset{\circ}{H}{}^1(y^\alpha,\Omega)$ that the seminorm $\|\nabla(\cdot)\|_{L^2(y^\alpha,\mathcal{C})}$ is equivalent to the norm $\|\cdot\|_{\mathcal{C}}$. Then

$$\|\mathcal{U}-\mathcal{U}_M\|_{\mathcal{C}} \lesssim \left\|\nabla\left(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right)\right\|_{L^2(y^\alpha,\mathcal{C})}$$

Since $\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}$ vanishes outside $\mathcal{C}_{\mathcal{Y}}$ we have

$$\left\|\nabla\left(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right)\right\|_{L^2(y^\alpha,\mathcal{C})} \leq \left\|\nabla\left(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right)\right\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} + \|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})}. \tag{39}$$

From [6, eq. (5.8)] we have that the second term on the right hand side of (39) is exponentially small in $\mathcal{Y}$, in fact,

$$\|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} \lesssim e^{-\gamma\mathcal{Y}/2}\|f\|_{H^{-s}(\Omega)}.$$

Taking into account that

$$\mathcal{Y} = c\log M$$

with $c \geq 2/\gamma$ we obtain

$$\|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} \lesssim \frac{1}{M}\|f\|_{H^{-s}(\Omega)}. \tag{40}$$

Now we consider the first term of the right hand side of (39). We have

$$\begin{aligned}
&\|\nabla(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \\
&\qquad\leq \|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} + \|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \\
&\qquad =: A + B,
\end{aligned} \tag{41}$$

where we have used that $\nabla_{x'}\left(\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right) = \pi^1_{y,\{\mathcal{Y}\}}\left(\nabla_{x'}\mathcal{U}\right)$.

We can estimate A as follows. On I_1, using the first inequality of (35) with $v = \nabla_{x'}\mathcal{U}$ we have

$$\|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,I_1)} \leq C|I_1|^\beta\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)}$$

for $\beta \geq 0$. Taking $\beta = 1-\sigma$, and since

$$|I_1| = c\log(M)\left(\frac{1}{M}\right)^{\frac{1}{1-\sigma}},$$

we obtain

$$\|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,I_1)} \leq C(\log M)^{1-\sigma}\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_1)}. \tag{42}$$

On intervals $I_i, i = 2,\ldots,M-1$, using the first inequality of (34) we have

$$\|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{0,I_i} \leq C|I_i|\|\partial_y\nabla_{x'}\mathcal{U}\|_{0,I_i},$$

and, taking into account (23) and (24), we obtain

$$\|\nabla_{x'}\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,I_i)} \leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_i)}.$$

Finally, for I_M, if π^1_y is the interpolation operator on the grid $\mathcal{G}^M$ defined like $\pi^1_{y,\{\mathcal{Y}\}}$ but without imposing $\pi^1_y(\cdot)(y_M) = 0$ (that is, $\pi^1_y(v)(y_M) = v(y_M)$), we have

$$\begin{aligned}\left\|\nabla_{x'}\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} &\leq \left\|\nabla_{x'}\mathcal{U} - \pi^1_y\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \\ &\quad + \left\|\left(\pi^1_y - \pi^1_{y,\{\mathcal{Y}\}}\right)\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)}\end{aligned} \tag{43}$$

The first term can be bounded in the same way as for $I_i, 2 \leq i \leq M-1$, obtaining

$$\left\|\nabla_{x'}\mathcal{U} - \pi^1_y\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_M)},$$

and for the second one we have

$$\left\|\left(\pi^1_y - \pi^1_{y,\{\mathcal{Y}\}}\right)\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \leq C\,(\log M)^{\frac{\alpha}{2}}\,|I_M|^{\frac{1}{2}}\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_{L^2(\Omega)},$$

since $\pi^1_y - \pi^1_{y,\{\mathcal{Y}\}}$ is a linear function in the variable y with values in $L^2(\Omega)$ vanishing at $y = y_{M-1}$. Inserting the previous inequalities into (43) we obtain

$$\begin{aligned}\left\|\nabla_{x'}\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} &\leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_M)} \\ &\quad + (\log M)^{\frac{\alpha}{2}}\,|I_M|^{\frac{1}{2}}\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_0.\end{aligned} \tag{44}$$

Using [6, eq. (A.10) and Lemma 16], we have

$$\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_0 \leq \mathcal{Y}^{-\frac{\alpha}{2}-1+\beta}e^{-\mathcal{Y}\gamma/2}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,\gamma},\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}$$

and since $|I_M| \leq C(\log M)\frac{1}{M}\mathcal{Y}^\sigma$ it results

$$\begin{aligned}&(\log M)^{\frac{\alpha}{2}}\,|I_M|^{\frac{1}{2}}\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_0 \leq \\ &\quad C\,(\log M)^{\frac{1+\alpha}{2}}\left(\frac{1}{M}\right)^{\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-1+\beta+\frac{\sigma}{2}}e^{-\mathcal{Y}\gamma/2}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,\gamma},\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}.\end{aligned} \tag{45}$$

Taking again $\beta = 1-\sigma$ and since

$$\max\left\{1-\sigma,\, 1+\frac{\sigma\alpha}{2}, \frac{1+\alpha}{2}\right\} = 1+\frac{\sigma\alpha}{2}$$

we have from inequalities (42)-(45) that

$$A \leq C(\log M)^{1+\frac{\sigma\alpha}{2}}\Bigg[\frac{1}{M}\|\partial_y \nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_\mathcal{Y})} + \left(\frac{1}{M}\right)^{\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha+\sigma}{2}}e^{-\mathcal{Y}\gamma/2}\|\partial_y \nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}\Bigg]. \quad (46)$$

It remains to estimate the term B in equation (41). Using the second interpolation error estimate from (35) it follows

$$\begin{aligned}\|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} &\leq |I_1|^\beta\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)} \\ &\leq C(\log M)^{1-\sigma}\tfrac{1}{M}\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_1)}\end{aligned} \quad (47)$$

if $\beta = 1-\sigma$. On intervals $I_i, i = 2,\dots,M-1$ using again the standard error estimates (34) and properties (23)–(24) of $\mathcal{G}^M$, we obtain

$$\|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,I_i)} \leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_i)}. \quad (48)$$

For the interval I_M we have again (recall the definition of $\pi^1_\mathcal{Y}$ before equation (43))

$$\begin{aligned}\|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,I_M)} &\leq \|\partial_y(\mathcal{U}-\pi^1_y\mathcal{U})\|_{L^2(y^\alpha,I_M)} + \\ &\qquad \left\|\partial_y\left[\left(\pi^1_y-\pi^1_{y,\{\mathcal{Y}\}}\right)\mathcal{U}\right]\right\|_{L^2(y^\alpha,I_M)} \\ &\leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_M)} + \\ &\qquad C\,(\log M)^{\frac{\alpha}{2}}\,|I_M|^{-\frac{1}{2}}\|\mathcal{U}(\cdot,\mathcal{Y})\|_{0,\Omega}.\end{aligned}$$

Then, since $|I_M| \gtrsim \frac{1}{M}$, and taking again [6, eq. (A.10) and Lemma 16] into account, it follows

$$(\log M)^{\frac{\alpha}{2}}\,|I_M|^{-\frac{1}{2}}\|\mathcal{U}(\cdot,\mathcal{Y})\|_0 \leq (\log M)^{\frac{\alpha}{2}}\left(\frac{1}{M}\right)^{-\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-1+\beta}e^{-\mathcal{Y}\gamma/2}\|\partial_y\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,\gamma},\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}. \quad (49)$$

Hence, from inequalities (47)–(49) with $\beta = 1 - \sigma$ we obtain

$$B \leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y^2 \mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} + \\ C\,(\log M)^{\frac{\alpha}{2}}\left(\frac{1}{M}\right)^{-\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-\sigma}e^{-\mathcal{Y}\gamma/2}\|\partial_y \mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})}. \quad (50)$$

Inserting (46) and (50) into (41) we have

$$\|\nabla(\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \leq \\ C(\log M)^{1+\frac{\alpha\sigma}{2}}\Bigg[\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} + \frac{1}{M}\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} + \\ \left(\frac{1}{M}\right)^{\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha+\sigma}{2}}e^{-\mathcal{Y}\gamma/2}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} + \\ \left(\frac{1}{M}\right)^{-\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-\sigma}e^{-\mathcal{Y}\gamma/2}\|\partial_y\mathcal{U}\|_{\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}}}\Bigg] \quad (51)$$

Since we take σ verifying (19), that is

$$\frac{\alpha+1}{2} < \sigma < 1$$

then we have from (36) that

$$\begin{aligned} \|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} &\lesssim \|f\|_{\mathbb{H}^{-s+1-\sigma}(\Omega)} \leq \|f\|_0, \\ \|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} &\lesssim \|f\|_{\mathbb{H}^{-s+1-\sigma}(\Omega)} \leq \|f\|_0, \\ \|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} &\lesssim \|f\|_{\mathbb{H}^{-s+1-\sigma}(\Omega)} \leq \|f\|_0, \end{aligned}$$

and since for a fixed $\gamma_0 > 0$ it holds $y^{2\sigma} \leq Ce^{\gamma_0 y}$ for all $y \geq 1$ we also have

$$\|\partial_y\mathcal{U}\|^2_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} = \int_{\mathcal{Y}}^\infty \|\partial_y\mathcal{U}\|_0^2\,y^{\alpha+2\sigma}e^{\gamma y}\,dy \\ \lesssim \int_{\mathcal{Y}}^\infty \|\partial_y\mathcal{U}\|_0^2\,y^{\alpha}e^{(\gamma+\gamma_0)y}\,dy \lesssim \|f\|^2_{\mathbb{H}^{-s}(\Omega)} \leq \|f\|_0^2$$

if γ_0 is taken such that $0 \leq \gamma + \gamma_0 < 2\sqrt{\lambda_1}$. Then from (51) we have

$$\|\nabla(\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \leq C(\log M)^{1+\frac{\alpha\sigma}{2}} \left\{ \frac{1}{M} + \left(\frac{1}{M}\right)^{\frac{1}{2}} \mathcal{Y}^{-\frac{\alpha+\sigma}{2}} e^{-\mathcal{Y}\gamma/2} \right.$$
$$\left. + \left(\frac{1}{M}\right)^{-\frac{1}{2}} \mathcal{Y}^{-\frac{\alpha}{2}-\sigma} e^{-\mathcal{Y}\gamma/2} \right\} \|f\|_0.$$

Now, we need to consider that

$$\mathcal{Y} = c \log M$$

with $c > \frac{3}{\gamma}$ in order to obtain

$$\|\nabla(\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \leq C(\log M)^{\frac{3+\alpha\sigma}{2}} \frac{1}{M} \|f\|_0. \tag{52}$$

Inequality (39), together with (40) and (52), give the result. $\square$

5. Superconvergent approximations of a reaction–diffusion equation using graded meshes

The goal of this section is to prove superconvergence results for the standard $\mathcal{Q}_1$ finite element approximation of the reaction–diffusion model problem introduced below when appropriate graded meshes are used.

We consider the model problem

$$\begin{aligned} -\varepsilon^2 \Delta w + b(x) w &= f \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{53}$$

where $\Omega = (0,1)^2$, $0 < \varepsilon \ll 1$ is a small positive parameter, and

$$b(x_1, x_2) \geq 1 \text{ in } \Omega. \tag{54}$$

5.1. *Auxiliary results*

We will assume that $f \in \mathcal{C}^2([0,1]^2)$ and that it satisfies the compatibility conditions

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0. \tag{55}$$

It is known that under these hypotheses, the exact solution of problem (53) satisfies $w \in \mathcal{C}^4(\Omega) \cap \mathcal{C}^2(\overline{\Omega})$. Moreover, we have the following pointwise estimates for w and its derivatives (see [21, Lemma 4.1], [22]): if $0 \le k \le 4$ then

$$\left|\frac{\partial^k w}{\partial x_1^k}(x_1, x_2)\right| \le C\left(1 + \varepsilon^{-k} e^{-x_1/\varepsilon} + \varepsilon^{-k} e^{-(1-x_1)/\varepsilon}\right), \tag{56}$$

$$\left|\frac{\partial^k w}{\partial x_2^k}(x_1, x_2)\right| \le C\left(1 + \varepsilon^{-k} e^{-x_2/\varepsilon} + \varepsilon^{-k} e^{-(1-x_2)/\varepsilon}\right). \tag{57}$$

We also have some weighted a priori estimates for w which are uniform in the perturbation parameter ε (see [8, Lemma 3.1]): let $d(t) = \min\{t, 1-t\}$ be the distance to the boundary function on the interval $[0, 1]$, then

(i) if $0 \le k \le 4$, $\alpha + \beta \ge k - \frac{1}{2}, \alpha \ge 0, \beta > -\frac{1}{2}$, then

$$\varepsilon^\alpha \left\| d(x_1)^\beta \frac{\partial^k w}{\partial x_1^k} \right\|_0 \le C, \quad \varepsilon^\alpha \left\| d(x_2)^\beta \frac{\partial^k w}{\partial x_2^k} \right\|_0 \le C, \tag{58}$$

(ii) if $\alpha + \beta \ge \frac{5}{2}, \alpha \ge \frac{3}{4}, \beta > \frac{1}{2}$, then

$$\varepsilon^\alpha \left\| d(x_2)^\beta \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_0 \le C. \tag{59}$$

5.2. Finite element approximation on graded meshes

The standard weak formulation of problem (53) is: find $w \in H_0^1(\Omega)$ such that

$$\mathcal{B}(w, v) = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega),$$

where the bilinear form $\mathcal{B}$ is defined as

$$\mathcal{B}(w, v) = \int_\Omega (\varepsilon^2 \nabla w \cdot \nabla v + b w v) \, dx.$$

For a domain D, we will work with the ε-weighted H^1-norm (referred as ε-norm in what follows) defined by

$$\|v\|_{\varepsilon^2, D}^2 := \varepsilon^2 \|\nabla v\|_{0,D}^2 + \|v\|_{0,D}^2.$$

When $D = \Omega$, for simplicity, we drop the subscript Ω.

It is well known that under the hypothesis (54), the bilinear form $\mathcal{B}$ is uniformly continuous and coercive in the ε-norm, in particular

$$\|v\|_{\varepsilon^2}^2 \le \mathcal{B}(v, v) \quad \forall v \in H_0^1(\Omega).$$

In [7], an analysis for the approximation of problem (53) by bilinear finite elements using appropriate graded meshes was developed. Almost optimal convergence, uniform with respect to ε, was proven in that paper. The graded meshes used in [7], which depend on a parameter η, with $\frac{1}{2} < \eta < 1$, are constructed independently of the perturbation parameter ε. In [8], under the stronger restriction $\frac{3}{4} \leq \eta < 1$, supercloseness results for the same scheme considered in [7] were obtained. Specifically, the difference between the finite element solution and the Lagrange interpolant of the exact solution, in the ε-norm, is of higher order than the error itself. The constants in such estimates depend only weakly on the singular perturbation parameter. In this section, our aim is, starting from these known results, to obtain a higher order approximation by a local post-processing of the computed solution.

On $\Omega = (0,1)^2$, for $N \in \mathbb{N}$, $h = 1/N$, and a given grading parameter η we consider the mesh $\mathcal{T}_N$ introduced in Subsection 3.4. Associated with $\mathcal{T}_N$, we introduce the piecewise bilinear finite element space V_N defined by (25), and the finite element approximation $w_N \in V_N$ that solves

$$\mathcal{B}(w_N, v) = \int_\Omega f v \, dx \quad \forall v \in V_N. \tag{60}$$

5.3. A higher order approximation by post–processing

As is known, the supercloseness estimate (see [8, Theorem 4.7]):

$$\|w_N - w_I\|_{\varepsilon^2} \leq C h^2 \log^{\frac{1}{2}}(\tfrac{1}{\varepsilon}), \tag{61}$$

where $w_I \in V_N$ is the Lagrange interpolant of the exact solution w, can be used to improve the numerical approximation by a local post–processing.

Remark 5.1. *We note that the graded meshes over Ω used in [8] have been defined differently than in Subsection 3.4. However, the only properties of the mesh involved in the proof of inequality* (61) *are those given in Remark 3.5. Therefore, the inequality remains valid even with our definition of the meshes.*

We will define the post–processed w_N^* of the finite element solution w_N following [9, 23, 11]. We repeat the construction given in those papers for the sake of completeness. Since $\mathcal{T}_N$ is a tensor product mesh of a partition of $[0,1]$ with $2N$ subintervals, it can be viewed as a refinement of the coarser mesh $\mathcal{S}_N$, formed by elements S_{ij} with $1 \leq i, j \leq N$ as described in Figure 1.

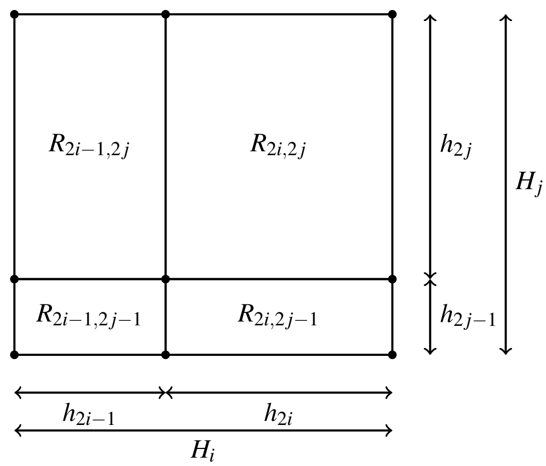

Figure 1: Element S_{ij}

Let I_2 be the biquadratic interpolation operator over the mesh $\mathcal{S}_N$, which for a function $v \in \mathcal{C}(\Omega)$, is defined on each S_{ij} as the Lagrange interpolant over the nine nodes indicated in Figure 1, i.e., the vertices of the four elements of the finer mesh $\mathcal{T}_N$ contained within S_{ij}. Consider

$$w_N^* := I_2 w_N.$$

Then we want to show that w_N^* is a second order approximation of w in the ε-norm.

We will need the following estimates for the operator I_2.

Lemma 5.1. *There exists a constant C such that, for any $v \in V_N$, and $S_{ij} \in \mathcal{S}_N$ we have*

$$\|I_2 v\|_{L^\infty(S_{ij})} \leq C \|v\|_{L^\infty(S_{ij})}. \tag{62}$$

Proof. For $(x_1, x_2) \in S_{ij}$ and $\alpha, \beta \in \{1, 2, 3\}$ fixed, we define

$$\varphi_{\alpha\beta}(x_1, x_2) = \prod_{k \neq \alpha} \frac{x_1 - x_1^k}{x_1^\alpha - x_1^k} \prod_{\ell \neq \beta} \frac{x_2 - x_2^\ell}{x_2^\beta - x_2^\ell} \tag{63}$$

where (x_1^k, x_2^ℓ), with $k, \ell = 1, 2, 3$, are the interpolation nodes on S_{ij}. Then, we can write

$$I_2 v = \sum_{\alpha,\beta=1,2,3} v(x_1^\alpha, x_2^\beta)\, \varphi_{\alpha\beta}, \qquad \text{on } S_{ij}. \tag{64}$$

Setting H_i and H_j as the lengths of the element S_{ij} along the directions of the x_1 and x_2 axes respectively, as in Figure 1, and $h^{x_1}_{min} := \min\{h_{2i-1}, h_{2i}\}$, $h^{x_2}_{min} := \min\{h_{2j-1}, h_{2j}\}$, we have that $|x_1 - x_1^k| \le H_i$, $|x_2 - x_2^\ell| \le H_j$, $|x_1^\alpha - x_1^k| \ge h^{x_1}_{min}$ and $|x_2^\beta - x_2^\ell| \ge h^{x_2}_{min}$.

Therefore, we obtain

$$\|\varphi_{\alpha\beta}\|_{L^\infty(S_{ij})} \le \frac{H_i^2 H_j^2}{(h^{x_1}_{min})^2 (h^{x_2}_{min})^2} \le C,$$

where in the last inequality we used that the ratios $H_i/h^{x_1}_{min}$, $H_j/h^{x_2}_{min}$ are uniformly bounded because the ratios h_{i+1}/h_i, h_{j+1}/h_j are as well (see Remark 3.5).

Summing up, we conclude that

$$||I_2 v||_{L^\infty(S_{ij})} \le \sum_{\alpha,\beta=1,2,3} ||\varphi_{\alpha\beta}||_{L^\infty(S_{ij})} ||v||_{L^\infty(S_{ij})} \le C||v||_{L^\infty(S_{ij})}$$

as we wanted to prove. □

Lemma 5.2. *There exists a constant C, such that, for any $v \in V_N$,*

$$\left\| \frac{\partial I_2 v}{\partial x_1} \right\|_{L^\infty(S_{ij})} \le C \left\| \frac{\partial v}{\partial x_1} \right\|_{L^\infty(S_{ij})},$$

$$\left\| \frac{\partial I_2 v}{\partial x_2} \right\|_{L^\infty(S_{ij})} \le C \left\| \frac{\partial v}{\partial x_2} \right\|_{L^\infty(S_{ij})}.$$

Proof. Let us prove the first inequality. Clearly, analogous arguments apply to obtain the second one.

Using expressions (64) for $I_2 v$ and (63) for $\varphi_{\alpha\beta}$, we obtain that

$$\begin{aligned}\frac{\partial I_2 v}{\partial x_1}(x_1, x_2) &= \sum_{\beta=1,2,3} \sum_{\alpha=1,2,3} \frac{\partial \varphi_{\alpha\beta}}{\partial x_1}(x_1, x_2) v(x_1^\alpha, x_2^\beta) \\ &= \sum_{\beta=1,2,3} \prod_{\ell \ne \beta} \frac{x_2 - x_2^\ell}{x_2^\beta - x_2^\ell} \sum_{\alpha=1,2,3} \frac{\partial}{\partial x_1} \left(\prod_{k \ne \alpha} \frac{x_1 - x_1^k}{x_1^\alpha - x_1^k} \right) v(x_1^\alpha, x_2^\beta).\end{aligned}$$

We observe that the ratios involving the x_2 variable can be bounded by a constant, as in the previous proof. On the other hand, for each β, let $p_\beta(x_1)$ be the quadratic interpolant of $v_\beta := v(\cdot, x_2^\beta)$ over the points x_1^α, $\alpha = 1, 2, 3$, that is

$$p_\beta(x_1) = \sum_{\alpha=1,2,3} \prod_{k \ne \alpha} \frac{x_1 - x_1^k}{x_1^\alpha - x_1^k} v(x_1^\alpha, x_2^\beta)$$

and therefore

$$\frac{\partial I_2 v}{\partial x_1}(x_1,x_2) = \sum_{\beta=1,2,3}\prod_{\ell\neq\beta}\frac{x_2 - x_2^\ell}{x_2^\beta - x_2^\ell}p'_\beta(x_1).$$

Since we can also write

$$p_\beta(x_1) = v_\beta(x_1^1) + \tfrac{v_\beta(x_1^2)-v_\beta(x_1^1)}{x_1^2-x_1^1}(x_1 - x_1^1) \\ + \frac{\tfrac{v_\beta(x_1^3)-v_\beta(x_1^2)}{x_1^3-x_1^2} - \tfrac{v_\beta(x_1^2)-v_\beta(x_1^1)}{x_1^2-x_1^1}}{x_1^3-x_1^1}(x_1 - x_1^1)(x_1 - x_1^2)$$

we have, if $x_1^M = \frac{x_1^1+x_1^2}{2}$, that

$$p'_\beta(x_1) = \tfrac{v_\beta(x_1^2)-v_\beta(x_1^1)}{x_1^2-x_1^1} + 2\frac{\tfrac{v_\beta(x_1^3)-v_\beta(x_1^2)}{x_1^3-x_1^2} - \tfrac{v_\beta(x_1^2)-v_\beta(x_1^1)}{x_1^2-x_1^1}}{x_1^3-x_1^1}\left(x_1 - x_1^M\right).$$

Afterwards, by the Mean Value Theorem, there exist $\zeta_0 \in (x_1^1, x_1^2)$ and $\zeta_1 \in (x_1^2, x_1^3)$ such that

$$p'_\beta(x_1) = v'_\beta(\zeta_0) + 2\tfrac{v'_\beta(\zeta_1)-v'_\beta(\zeta_0)}{x_1^3-x_1^1}(x_1 - x_1^M).$$

Now, remembering that $|x_1^3 - x_1^1| = H_i$ and $|x_1 - x_1^M| \leq H_i$, we get

$$|p'_\beta(x_1)| \leq |v'_\beta(\zeta_0)| + 2\tfrac{|v'_\beta(\zeta_1)|+|v'_\beta(\zeta_0)|}{H_i}H_i \leq 5\left\|\tfrac{\partial v}{\partial x_1}(\cdot, x_2^\beta)\right\|_{L^\infty(x_1^1,x_1^3)}.$$

Summing up, we obtain

$$\left\|\tfrac{\partial I_2 v}{\partial x_1}\right\|_{L^\infty(S_{ij})} \leq C\sum_{\beta=1,2,3}\left\|\tfrac{\partial v}{\partial x_1}(\cdot, x_2^\beta)\right\|_{L^\infty(x_1^1,x_1^3)} \leq C\left\|\tfrac{\partial v}{\partial x_1}\right\|_{L^\infty(S_{ij})}$$

as we wanted to show. □

Lemma 5.3. *Let w be the solution of* (53)*, and $I_2 w$ its piecewise biquadratic interpolation on $\mathcal{S}_N$. There exists a constant C such that*

$$\|w - I_2 w\|_{\varepsilon^2} \leq Ch^2. \tag{65}$$

Proof. The result follows from the a priori estimates provided in Section 5.1 and the following interpolation error estimates for the operator I_2 (see [24,

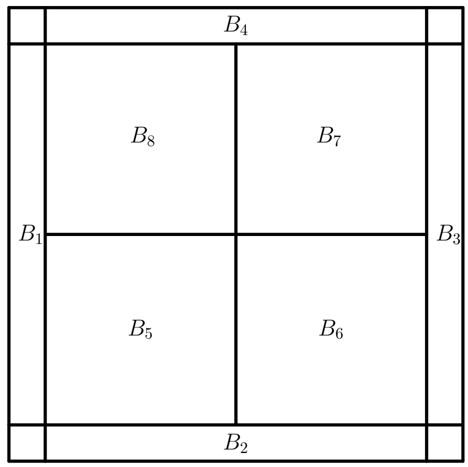

Figure 2: Split of the unitary square domain used in the proof of Lemma 5.3

Theorem 2.7] and [9, Lemma 4.1]). For $v \in H^3(S_{ij})$, we have

$$\|v - I_2 v\|_{0,S_{ij}} \leq C\left[H_i^2 \left\|\frac{\partial^2 v}{\partial x_1^2}\right\|_{0,S_{ij}} + H_j^2 \left\|\frac{\partial^2 v}{\partial x_2^2}\right\|_{0,S_{ij}}\right], \tag{66}$$

$$\left\|\frac{\partial(v - I_2 v)}{\partial x_1}\right\|_{0,S_{ij}} \leq C\left[H_i^2 \left\|\frac{\partial^3 v}{\partial x_1^3}\right\|_{0,S_{ij}} + H_j^2 \left\|\frac{\partial^3 v}{\partial x_1 \partial x_2^2}\right\|_{0,S_{ij}}\right], \tag{67}$$

$$\left\|\frac{\partial(v - I_2 v)}{\partial x_2}\right\|_{0,S_{ij}} \leq C\left[H_i^2 \left\|\frac{\partial^3 v}{\partial x_1^2 \partial x_2}\right\|_{0,S_{ij}} + H_j^2 \left\|\frac{\partial^3 v}{\partial x_2^3}\right\|_{0,S_{ij}}\right], \tag{68}$$

where the constant C is independent of the element S_{ij} and v. With the notation introduced in Figure 2, we write Ω as

$$\Omega = \bigcup_{i=1}^{8} B_i,$$

where

$$B_1 = \bigcup_{j=1}^{N} S_{1j}, \quad B_2 = \bigcup_{i=1}^{N} S_{i1}, \quad B_3 = \bigcup_{j=1}^{N} S_{Nj}, \quad B_4 = \bigcup_{i=1}^{N} S_{iN},$$

and

$$
\begin{aligned}
B_5 &= \bigcup \{S_{ij} : 2 \le i,\, j \le N/2 - 1\}\,, \\
B_6 &= \bigcup \{S_{ij} : N/2 \le i \le N-1,\, 2 \le j \le N/2 - 1\}\,, \\
B_7 &= \bigcup \{S_{ij} : N/2 \le i,\, j \le N - 1\}\,, \\
B_8 &= \bigcup \{S_{ij} : 2 \le i \le N/2-1,\, N/2 \le j \le N - 1\}\,.
\end{aligned}
$$

Due to the symmetry of the problem, it is sufficient to estimate (65) over B_1 and B_5. Starting with B_1, we have

$$
\|w - I_2 w\|^2_{\varepsilon^2,\, B_1} = \varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, B_1} + \|w - I_2 w\|^2_{0,\, B_1}. \tag{69}
$$

Here we obtain, for the first term on the right-hand side

$$
\varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, B_1} = \varepsilon^2 \left\| \tfrac{\partial(w - I_2 w)}{\partial x_1} \right\|^2_{0,\, B_1} + \varepsilon^2 \left\| \tfrac{\partial(w - I_2 w)}{\partial x_2} \right\|^2_{0,\, B_1}. \tag{70}
$$

From (56) and (57), we can observe that

$$
\left| \frac{\partial w}{\partial x_i} \right| \le \frac{C}{\varepsilon}, \qquad i = 1, 2.
$$

Taking this into account, using Lemma 5.2 and since $|B_1| = H_1 = Ch^{\frac{1}{1-\eta}}$, we obtain

$$
\varepsilon^2 \left\| \frac{\partial(w - I_2 w)}{\partial x_1} \right\|^2_{0,\, B_1} \le Ch^{\frac{1}{1-\eta}}.
$$

A similar result is obtained for the derivative with respect to x_2 of the interpolation error and, therefore, we deduce from (70) that

$$
\varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, B_1} \le Ch^{\frac{1}{1-\eta}}. \tag{71}
$$

Similarly, using Lemma 5.1 and taking into account that w is uniformly bounded, we have

$$
\|w - I_2 w\|^2_{0,\, B_1} \le Ch^{\frac{1}{1-\eta}}. \tag{72}
$$

Finally, with $\eta \ge \frac{3}{4}$ in (71) and (72), it follows

$$
\|w - I_2 w\|^2_{\varepsilon^2,\, B_1} \le Ch^4.
$$

Now we deal with the estimate on B_5. We consider S_{ij} for $2 \le i, j \le N/2-1$, then

$$\|w - I_2 w\|^2_{\varepsilon^2,\, S_{ij}} = \varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, S_{ij}} + \|w - I_2 w\|^2_{0,\, S_{ij}}. \tag{73}$$

We have again for the first term on the right side

$$\varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, S_{ij}} = \varepsilon^2 \left\| \frac{\partial(w-I_2 w)}{\partial x_1} \right\|^2_{0,\, S_{ij}} + \varepsilon^2 \left\| \frac{\partial(w-I_2 w)}{\partial x_2} \right\|^2_{0,\, S_{ij}}. \tag{74}$$

We note that the side lengths H_i, H_j of the elements S_{ij} considered here satisfy $H_i \le C h x_1^\eta$, $H_j \le C h x_2^\eta$ for $(x_1, x_2) \in S_{ij}$. Using this in equation (67) we have

$$\left\| \frac{\partial(w-I_2 w)}{\partial x_1} \right\|^2_{0,\, S_{ij}} \le C h^4 \left[\left\| x_1^{2\eta} \frac{\partial^3 w}{\partial x_1^3} \right\|^2_{0,\, S_{ij}} + \left\| x_2^{2\eta} \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|^2_{0,\, S_{ij}} \right] \le C h^4 \varepsilon^{-2}$$

where the last inequality is a consequence of (58), (59) and the condition $\eta \ge \frac{3}{4}$. Since the corresponding inequality

$$\left\| \frac{\partial(w-I_2 w)}{\partial x_2} \right\|^2_{0,\, S_{ij}} \le C h^4 \varepsilon^{-2}$$

is proved analogously, we obtain

$$\varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, S_{ij}} \le C h^4. \tag{75}$$

Similarly, for the second term on the right side of (73), using (66) and then (58), with $\eta \ge \frac{3}{4}$, we have

$$\|(w - I_2 w)\|^2_{0,\, S_{ij}} \le C h^4 \left[\left\| x_1^{2\eta} \frac{\partial^2 w}{\partial x_1^2} \right\|^2_{0,\, S_{ij}} + \left\| x_2^{2\eta} \frac{\partial^2 w}{\partial x_2^2} \right\|^2_{0,\, S_{ij}} \right] \le C h^4. \tag{76}$$

Finally, summing inequalities (75) and (76) over all $S_{ij} \in B_5$,

$$\|w - I_2 w\|^2_{\varepsilon^2, B_5} \le C h^4$$

concluding the proof. □

The proof of the next lemma follows from the same arguments used in [9, Lemma 4.2].

Lemma 5.4. *There exists a constant C such that, for any $v \in V_N$,*

$$\|I_2 v\|_{\varepsilon^2} \leq C\|v\|_{\varepsilon^2}. \tag{77}$$

Proposition 5.1. *Let w be the solution of* (53)*, $w_N \in V_N$ its finite element approximation, and $w_N^* = I_2 w_N$. Suppose that $\frac{3}{4} \leq \eta < 1$. Then, there exists a constant C such that*

$$\|w - w_N^*\|_{\varepsilon^2} \leq Ch^2 \log^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\right). \tag{78}$$

Proof. Since $I_2 w_I = I_2 w$, we have

$$\|w - w_N^*\|_{\varepsilon^2} \leq \|w - I_2 w\|_{\varepsilon^2} + \|I_2(w_I - w_N)\|_{\varepsilon^2},$$

and therefore, combining (65), (77) and (61), we conclude the proof. □

Remark 5.2. *Under the assumed regularity and compatibility conditions on f, the estimates* (56), (57), (58) *and* (59)*, also hold in the non-singularly perturbed case (moderate values of ε). Therefore, the result of superconvergence of Proposition 5.1 holds in that case as well. We will use this fact in the next Section to obtain our main result.*

Remark 5.3. *The proof of Proposition 5.1 relies heavily on the properties of graded meshes, which are a type of layer–adapted meshes. Superconvergence results for the approximation of singularly perturbed problems have also been obtained for ε–dependent layer–adapted meshes, such as piecewise–uniform Shishkin–type meshes (see [25] and the references therein), A–type meshes [26], and other ε–dependent graded meshes [27]. These meshes have a simple structure but require prior knowledge of the boundary layer locations.*

Following the discussion in [28, Remark 3], and motivated by the results in [29] on the use of equidistributed meshes for singularly perturbed parabolic systems, an interesting question is whether a similar superconvergence result can be established for adaptive or moving meshes. We hope to address this question in future work.

6. The error estimate

We recall the functions $U_i \in H_0^1(\Omega)$, associated with the eigenvalue μ_i, which are solutions of the variational problems (16), and their piecewise bilinear approximations $U_{i,N} \in V_N$ introduced in (26) with the space V_N defined in (25). Let $U^*_{i,N}$ be the post-processed version of $U_{i,N}$ introduced in the previous section. According to Proposition 5.1, with $\varepsilon = \sqrt{\mu_i}$ and $h = \frac{1}{N}$, it holds that

$$\|U_i - U^*_{i,N}\|_{\mu_i} \leq C\frac{1}{N^2}\,|\log \mu_i|^{\frac{1}{2}}\,(\log M)^s \tag{79}$$

where C depends on f and s, and is independent of N, M, and μ_i. We used estimate $|v_i(0)| \leq (\log M)^s$ from Remark 3.4.

Now we choose $N = M^{\frac{3}{4}}$. Then, by inserting (37) (absorbing $\|f\|_0$ into the constant C) and (33) into (32), and using (79) for $i = 1, 2, \ldots, M$, we obtain

$$\begin{aligned}
\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} &\lesssim (\log M)^{\frac{3+\sigma\alpha}{2}}\,M^{-1} + \left(\sum_{i=1}^{M} |\log \mu_i|(\log M)^{2s} N^{-4}\right)^{\frac{1}{2}} \\
&\leq CM^{-1}\left[(\log M)^{\frac{3+\sigma\alpha}{2}} + (\log M)^s\left(\max_{1\leq i\leq M} |\log \mu_i|\right)^{\frac{1}{2}}\right] \\
&\leq CM^{-1}\left[(\log M)^{\frac{3+\sigma\alpha}{2}} + (\log M)^{\frac{1}{2}+s}\right] \\
&\leq CM^{-1}\,(\log M)^{\max\left(\frac{3+\sigma\alpha}{2},\frac{1}{2}+s\right)}
\end{aligned} \tag{80}$$

where we used, taking into account the upper and lower bounds (17) and (18) for the eigenvalues μ_i, that

$$\max_{1\leq i\leq M} |\log \mu_i| \leq C \log M.$$

Then we have proven our main theorem, which we can now state.

Theorem 6.1. *Assume that $f \in \mathcal{C}^2(\overline{\Omega})$ is a function satisfying the compatibility condition* (55)*, with $\Omega = (0,1)^2$. Let $s \in (0,1)$ and u be the solution of problem* (1)*. Given $M \in \mathbb{N}$, let $N = M^{\frac{3}{4}}$. We consider the approximation $u_{M,N}$ given by* (28)*, with the grid $\mathcal{G}^M$ introduced in Section 3.4 with the parameter σ satisfying* (19) *and the graded mesh $\mathcal{T}_N$ defined with η satisfying* (21)*. Then there exists a constant C, depending only on s and f, such that*

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C\,(\log M)^t\,M^{-1}, \tag{81}$$

with $t = \max\left\{\frac{3+\sigma\alpha}{2}, \frac{1}{2} + s\right\}$.

We can rewrite the result in terms of the total number N_{dof} of degrees of freedom. Notice that our discretization requires solving M reaction–diffusion problems, each one of them having $O(N^2)$ degrees of freedom. It follows that $N_{dof} \sim M^{\frac{5}{2}}$. Therefore, we can rewrite (81) as

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C \left(\log N_{dof}\right)^t N_{dof}^{-\frac{2}{5}}.$$

This order of convergence is suboptimal, since for a regular two–dimensional problem an error of order $N_{dof}^{-\frac{1}{2}}$ is expected. However, it is slightly better than the result obtained in [6, Theorem 3] where, additionally, a stronger boundary compatibility condition is additionally assumed on f. On the other hand, we would like to emphasize that the M linear systems coming from the approximation of the reaction–diffusion problems (26) have a simple structure, they are of the form

$$\mu_i A_1 + A_0 = d_s v_i(0) b \qquad i = 1, \ldots, M$$

with matrices A_0 and A_1, and the vector b, depending only on the graded mesh $\mathcal{T}_N$. Then M linear systems can be obtained simultaneously once the eigenpairs (μ_i, v_i) are known, and parallelization algorithms could be applied to improve performance.

7. Numerical examples

In this section, we present numerical examples to validate the theoretical results of the main Theorem 6.1 regarding the approximation of problem (1). Additionally, we provide a numerical example that confirms the result of Proposition 5.1, demonstrating the robust convergence of the post–processed solution for reaction–diffusion problems with respect to the singular perturbation parameter.

7.1. Approximation of the fractional model problem

In order to confirm the results of Theorem 6.1, we approximate the solution of the problem

$$\begin{aligned} (-\Delta)^s u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{82}$$

in two examples. The calculations were implemented using GNU Octave [30]. The eigenvalue problems of Section 3.3 were solved using the command `eig`, and the linear problems (26) were solved using Octave's *backslash* operator "\". Taking into account the symmetry of the problems, errors are computed in the subdomain $[0, \frac{1}{2}]$ in the one-dimensional case and in $[0, \frac{1}{2}]^2$ in the two-dimensional case.

We measure the error in the energy norm $\|\cdot\|_{\mathbb{H}^s(\Omega)}$, which is estimated by

$$\int_\Omega |f(u - u_{M,N})|$$

since

$$\|u - u_{M,N}\|^2_{\mathbb{H}^s(\Omega)} \lesssim \|\mathcal{U} - \mathcal{U}_{M,N}\|^2_{L^2(y^\alpha,\mathcal{C})} = d_s \int_\Omega f(u - u_{M,N}).$$

Example 1. We consider problem (82) with $\Omega = (0,1)^2$ and

$$f(x,y) = (x+y)(x+y-2)((x-y)^2-1).$$

For the exponents $s = 0.25$ and $s = 0.75$, we approximate the problem as stated in Section 3.4 with the following parameters.

- $\sigma = \frac{1-0.9s}{1+0.1s}$, which implies $\frac{1}{1-\sigma} = \frac{1}{s} + 0.1$,
- $\mathcal{Y} = 2\log M$,
- η varies in $\{0.8, 0.85, 0.9\}$,
- M is taken as $M = 2^i$, with $i = 4, 5, \ldots, 9$.

Since the exact solutions are unknown, the numerical errors were estimated comparing with a solution obtained for the highest value of $M = 1024$.

We show plots of the solutions in Figure 3 obtained for $M = 256$. In Figure 4, we plot the estimated error in the $\mathbb{H}^s$–norm versus M on a logarithmic scale, observing a convergence order close to 1, which confirms the results of Theorem 6.1.

Furthermore, in order to check that some problems (16) become singularly perturbed, we report l_{min} (defined in Section 3.3) and $\mu_{min} := \min\{\mu_i : i = 1, \ldots, M\}$ in Table 7.1.

Example 2. In this example we take a right–hand side f which does not satisfy the compatibility condition (55), in two cases:

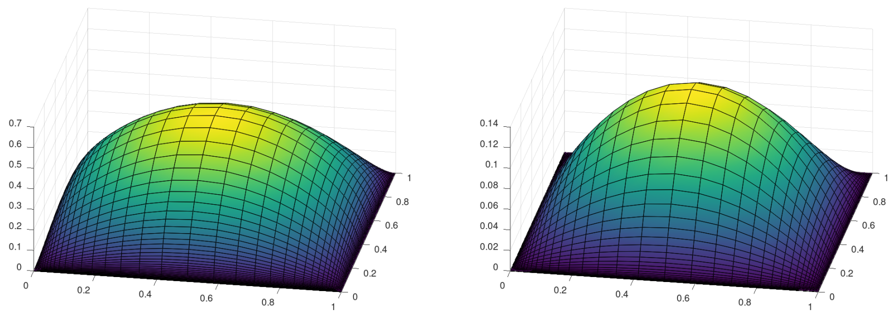

Figure 3: Numerical solutions for Example 1 with $s = 0.25$ (left) and $s = 0.75$ (right)

N	$s = 0.25$		$s = 0.75$	
	l_{min}	μ_{min}	l_{min}	μ_{min}
2^4	6.4124e-05	9.0244e-10	0.10423	0.001577
2^5	4.6742e-06	4.795e-12	0.048244	0.00033782
2^6	3.2709e-07	2.3481e-14	0.021436	6.6696e-05
2^7	2.2253e-08	1.0868e-16	0.0092602	1.2446e-05
2^8	1.4831e-09	4.8272e-19	0.0039186	2.2288e-06
2^9	9.7295e-11	2.0776e-21	0.0016323	3.8675e-07

Table 1: Report of l_{min} and μ_{min} for data of Example 1.

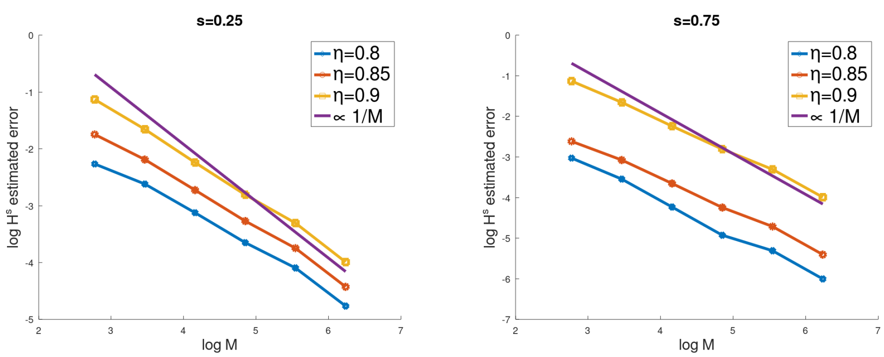

Figure 4: Numerical errors for Example 1

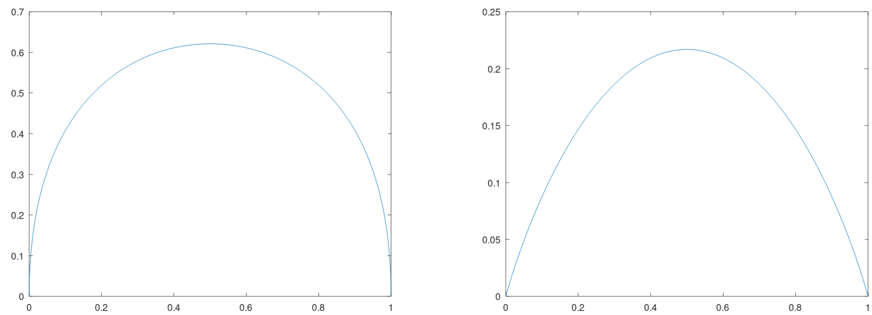

Figure 5: Numerical solutions for Example 2, 1d–case, with $s = 0.25$ (left) and $s = 0.75$ (right)

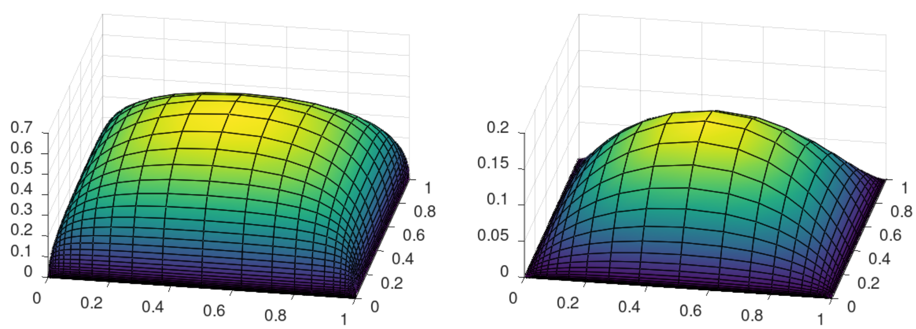

Figure 6: Numerical solutions for Example 2, 2d–case, with $s = 0.25$ (left) and $s = 0.75$ (right)

- the 1d–case: $\Omega = (0, 1)$, and $f(x) = 1$,
- the 2d–case: $\Omega = (0, 1)^2$, and $f(x, y) = 1$.

Figures 5 and 6 show plots of the solutions obtained for $M = 256$.

In Figures 7 and 8, we plot the estimated $\mathbb{H}^s$ versus M on a logarithmic scale. We observe that, although this example is not covered by the theory, the results are consistent with Theorem 6.1. We also show, in Figure 9, the errors obtained when using uniform meshes for the discretization of the singularly perturbed reaction–diffusion problems in the 2d–case, with the same number of elements as in the corresponding graded mesh cases. In this case, we observe that the order of convergence is reduced, becoming closer to $\frac{1}{2}$.

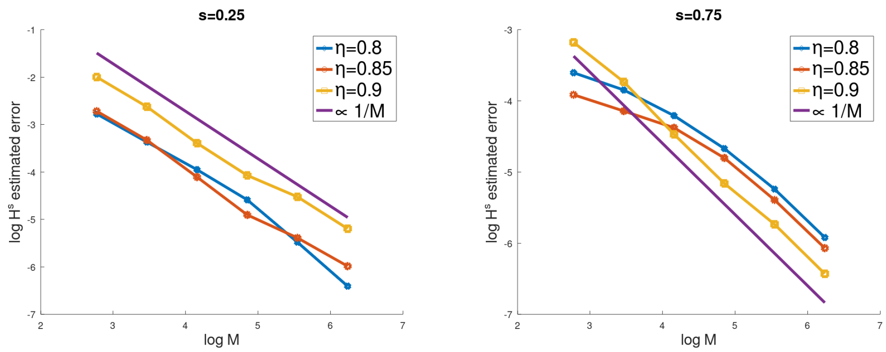

Figure 7: Numerical errors for Example 2, 1d–case

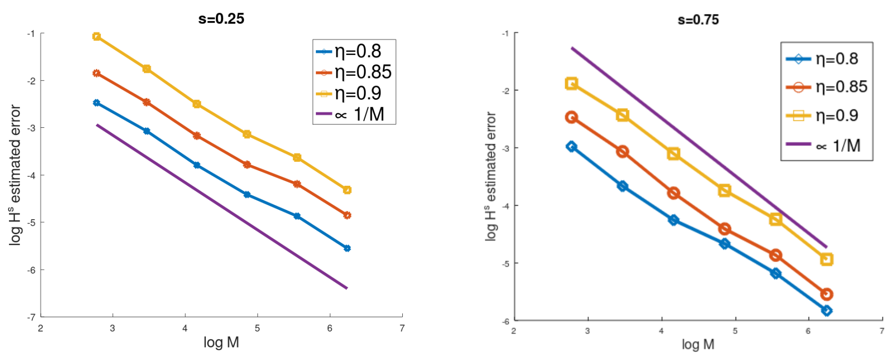

Figure 8: Numerical errors for Example 2, 2d–case

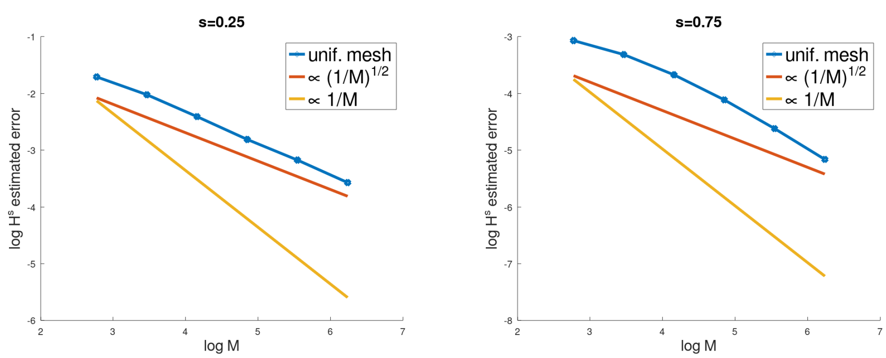

Figure 9: Numerical errors for Example 2, 2d–case, using uniform meshes on Ω

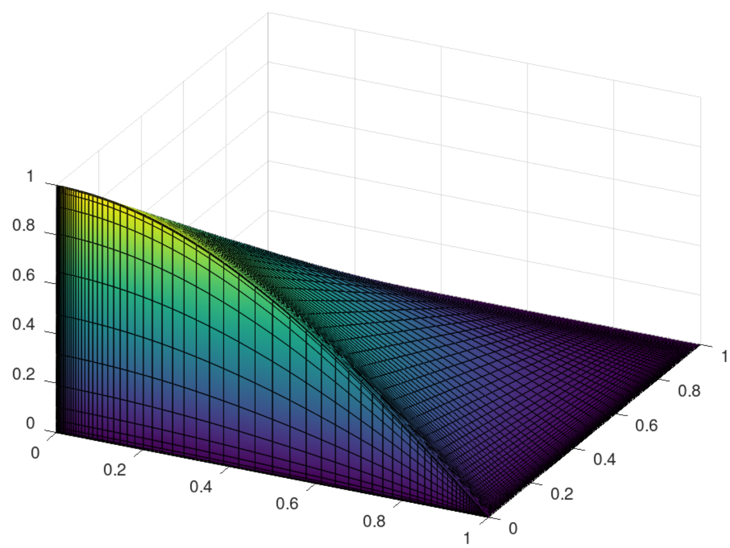

Figure 10: Typical solution of problem (7.2)

7.2. Approximation of reaction–diffusion problems by post–processing

We consider the singularly perturbed reaction–diffusion problem

$$-\varepsilon^2 \Delta w + w = f \quad \text{in } \Omega = (0,1)^2, \qquad w = 0 \quad \text{on } \partial\Omega,$$

where f is chosen so that the solution is

$$w(x,y) = \left(\cos\left(\frac{\pi}{2}x\right) - \frac{e^{-\frac{x}{\varepsilon}} - e^{\frac{1}{\varepsilon}}}{1 - e^{\frac{1}{\varepsilon}}} \right) \left(1 - y - \frac{e^{-\frac{y}{\varepsilon}} - e^{\frac{1}{\varepsilon}}}{1 - e^{\frac{1}{\varepsilon}}} \right).$$

This is a widely used test problem in the literature (see, e.g. [31, 32]). Its solution exhibits only two layers, near $x = 0$ and $y = 0$. It is interesting to note that estimates from equations (56) and (57) are sharp for the solution in the region influenced by the layers, while the solution becomes smoother further away from them. Thus, the example is sufficiently typical to verify the results of Proposition 5.1. A plot of the solution, obtained for $\varepsilon = 10^{-4}$, is shown in Figure 10

For ε varying between 10^{-4} and 10^{-10} we compute the finite element solution w_N of the discrete variational formulation (60) as described in Section

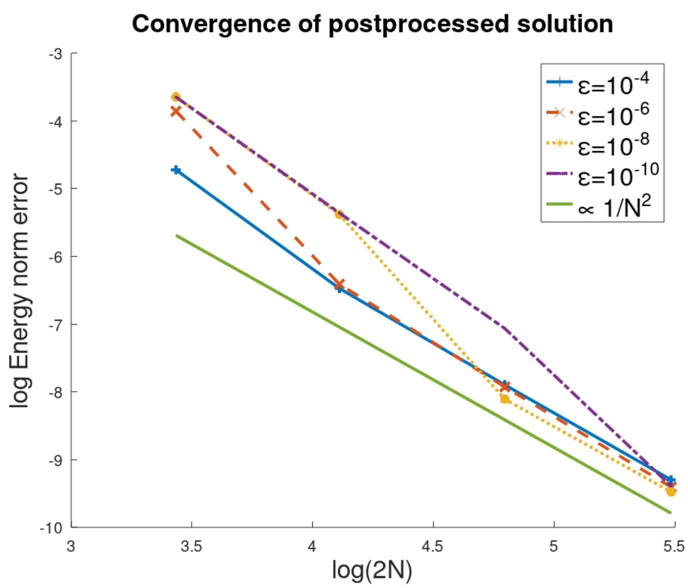

Figure 11: Superconvergence in the energy norm of the post–processed solution for example in Section 7.2

5.2, using graded meshes $\mathcal{T}_N$, with $N = 15, 30, 60, 120$. Then, we compute the post–processed w_N^* introduced in Section 5.3. In Figure 11, we plot (in logarithmic scale) the errors $\|w - w_N^*\|_{\varepsilon^2}$ against the number $2N$ of intervals along the x and y axes of the graded meshes. We observe that the errors are quite similar for all values of the singular perturbation parameter. The order of convergence with respect to N observed is quadratic in $h = \frac{1}{N}$ in agreement with Proposition 5.1.

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