A finite element discretization of fractional problems using graded meshes

Melani Barrios[a,b], Ariel L. Lombardi[a,b], Cecilia Penessi[a,b]

[a] *Departamento de Matemática, Escuela de Ciencias Exactas y Naturales, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario. Av. Pellegrini 250, S2000BTP Rosario, Argentina*
[b] *CONICET, Av. Pellegrini 250, S2000BTP Rosario, Argentina*

Abstract

In this paper, we deal with the finite element approximation of a homogeneous Dirichlet problem for fractional powers of symmetric second-order elliptic operators on a two-dimensional domain Ω. We employ the diagonalization technique introduced in [Banjai, Melenk, Nochetto, Otárola, Salgado, Schwab, Foundations of Computational Mathematics (2019) 19: 901–962], which proposes a semi-discretization in the extended variable of a truncated Caffarelli–Silvestre extension. This approach decouples the problem into the solution of independent second-order reaction-diffusion equations in Ω, several of which become singularly perturbed. For the case where $\Omega = (0,1)^2$, we propose to approximate all the decoupled problems by bilinear finite elements over a unique layer adapted, suitably graded, rectangular mesh, which can be designed independently of the eventual singular perturbation parameters. We prove the convergence of the proposed scheme and show numerical examples confirming the theoretical results.

Keywords: Non-local operators, Fractional Diffusion, Finite Element Method, Graded Meshes, Reaction–Diffusion equations, Singularly Perturbed Problems
2020 MSC: 65N30

---

1. Introduction

We are interested in finite element approximations of the Dirichlet problem for fractional powers of a symmetric second–order elliptic operator. Given a domain Ω in $\mathbb{R}^2$, a real $s \in (0,1)$, and a function f, the model problem reads as follows: find u solution of

$$\begin{aligned} \mathcal{L}^s u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

We consider, for simplicity, the operator $\mathcal{L}$ of the form

$$\mathcal{L}v = -\Delta v + \bar{c}(x)v, \tag{2}$$

with $\bar{c}(x) \in L^\infty(\Omega)$ a non-negative function defined on Ω. More generally, operators with a diffusion coefficient could also be considered.

The main difficulty in order to obtain efficient numerical methods for (1) is that $\mathcal{L}^s$ is a non-local operator [1, 2]. One of the most studied non-local operators is the Dirichlet Laplacian $\mathcal{L} = -\Delta$ due to its physical applications involving long–range or anomalous diffusion. For example, it is used in modeling the flow of certain particles in porous media (see [3]). Caffarelli and Silvestre, in [2], localize this problem by means of a non-uniformly elliptic PDE posed in one more spatial dimension. They showed that any power $s \in (0,1)$ of the fractional Laplacian in $\mathbb{R}^d$ can be realized as the Dirichlet-to-Neumann map of an extension to the upper half-space $\mathbb{R}^{d+1}_+$. This result was extended by Cabré and Tan [1] and by Stinga and Torrea [4] to consider bounded domains Ω and more general operators, thereby obtaining an extended problem posed on the semi-infinite cylinder $\mathcal{C} := \Omega \times (0, \infty)$.

Nochetto, Otárola and Salgado in [5], proposed to approximate the solution $u(x)$ of (1) by taking the trace $\mathfrak{u}(\cdot, 0)$ on $\Omega \times \{0\}$ of an approximation to the solution $\mathfrak{u}$ of the extended problem. Indeed, they analyzed the extended problem in the framework of weighted Sobolev spaces, and motivated by the rapid decay of $\mathfrak{u}$, they considered a truncation $\mathcal{C}_\mathcal{Y} = \Omega \times [0, \mathcal{Y}]$ of $\mathcal{C}$ and $\mathfrak{u}$ is approximated there by discretizing with first order tensor product finite elements. Subsequently, in [6], these authors, together with Banjai, Melenk and Schwab, extended the previous results in several directions. Particularly, they proposed a novel diagonalization technique which decouples the degrees of freedom introduced by a Galerkin (semi–)discretization in the extended variable. This technique reduces the y–semidiscrete Caffarelli–Silvestre extension to the solution of independent second–order reaction–diffusion equations posed on Ω, some of which are singularly perturbed. By introducing an hp finite element approximation of those reaction–diffusion problems, this decoupling allowed them to establish exponential convergence for analytic data f without assuming boundary compatibility.

This paper is mainly motivated by the observation that singularly perturbed reaction–diffusion problems on a square can be almost optimally approximated in the energy norm by an h version of piecewise bilinear finite elements using meshes (graded meshes) designed independently of the perturbation parameter [7, 8]. Then, we start with the diagonalization technique from [6], and propose a strategy to design a unique graded mesh on $\Omega = (0,1)^2$ to approximate the sequence of reaction–diffusion problems coming from the semi–discretization of the extended problem in the truncated cylinder $\mathcal{C}_{\mathcal{Y}}$. All those solutions are then combined as in [6] in order to obtain an approximation of the solution of (1). Our assumptions on the right hand side f are that it is $\mathcal{C}^2(\overline{\Omega})$ and it satisfies the compatibility condition (53) vanishing on the vertices of the square.

Our results are linear up to a logarithmic factor in the number of reaction–diffusion equations to be solved. However, it is important to study the approximation error in terms of the number of degrees of freedom. In particular, it follows from Theorem 5.1 that to obtain an error of almost order $O(M^{-1})$, we need to solve M reaction–diffusion problems, each one of them having $O(M^{\frac{3}{2}})$ degrees of freedom. Then, with a total number of $O(M^{\frac{5}{2}})$ degrees of freedom, we get an accuracy of $O(M^{-1})$ (up to logarithmic factors). This is slightly better than the complexity of the h version of the finite element method for a regular three dimensional problem. We notice that the discretization of the M reaction–diffusion problems leaves to M linear systems with matrices of the form $\mu_i A_1 + A_0$ and right–hand side $\zeta_i b$, with fixed matrices A_0 and A_1, and a fixed vector b. The coefficients μ_i and ζ_i are computed at the beginning of the process. Therefore, we think that this approach can be combined with suitable parallelization algorithms in order to obtain a better performance, but we do not delve into this issues in depth in this work.

The standard finite element methods for singularly perturbed problems produce very poor results when uniform or quasi–uniform meshes are used unless they are sufficiently refined. Consequently, these kind of meshes are not useful in practical applications, and therefore, several alternatives of appropriately adapted meshes have been considered in many papers. The best known are the Shishkin meshes (see [9, 10]). In this paper, we will consider graded meshes which were introduced in [7]. There, the authors have obtained almost optimal error estimates in the energy norm robust with respect to the singular perturbation parameter ε. It is interesting to note that, to achieve an expected bound of the energy norm error, those graded meshes can be defined independently of ε, a property which is exploited here to approximate the decoupled reaction–diffusion problems that appear with the approach consider in [6].

To obtain almost linear convergence in h to the solution of (1), our approach requires superlinear approximations of each one of the reaction–diffusion problems. In view of that, in Proposition 4.1, we show that a local post-processing of the bilinear finite element solution on graded meshes yields a superconvergent approximation that is almost uniform with respect to the singular perturbation parameter. We obtain that result as a consequence of a supercloseness property proved in [8] when the grading parameter defining the graded meshes is large enough. The technique to get robust superconvergence results has been previously used in [11], in the case of singularly perturbed convection–diffusion problems. Similar results for singularly perturbed convection–diffusion or reaction–diffusion on Shishkin meshes were obtained in [9, 10, 12, 13].

The rest of the paper is organized as follows. In Section 2, we introduce the model problem and its discretization which is based on the Caffarelli-Silvestre extension. Furthermore some auxiliary results are presented. Section 3 includes estimates of the semi-discretization error. In Section 4 we deal with the finite element approximation of singularly perturbed reaction–diffusion equations on graded meshes. In particular we show how a higher order approximation can be obtained from the computed solution by a simple local post-processing. Our main result concerning the error estimate is presented in Section 5, and finally, Section 6 contains some numerical experiments which confirm the theoretical results.

Throughout the paper, we adopt the following notation. For a domain D, we use standard notation for L^p and Sobolev spaces, as well as their respective norms and seminorms, namely,

$$\|u\|_{L^p(D)} := \left(\int_D |u|^p\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

$$\|u\|_{L^\infty(D)} := \inf\left\{C > 0 : |u(x)| \le C \; a.e.\right\},$$

$$\|u\|_{m,D} := \left\{\sum_{|\alpha|\le m} \|\mathcal{D}^\alpha u\|^2_{L^2(D)}\right\}^{1/2}, \quad |u|_{m,D} := \left\{\sum_{|\alpha|=m} \|\mathcal{D}^\alpha u\|^2_{L^2(D)}\right\}^{1/2}.$$

In particular, $\|u\|_{0,D}$ denotes the L^2-norm of u over D. When $D = \Omega$, and no confusion can arise, we will write $\|u\|_0$ instead of $\|u\|_{0,\Omega}$.

For a rectangle R, $\mathcal{P}_k(R)$ and $\mathcal{Q}_k(R)$ denote the spaces of polynomials of total degree less than or equal to k and polynomials of degree less than or equal to k in each variable, respectively, over R.

In addition, C will denote a constant that may depend on the fractional power s or the discretization parameters σ and η introduced in Subsection (2.4), and which is independent of the mesh sizes and of the singular parameters in reaction-diffusion problems. The value of C might change at each occurrence. The notation $a \lesssim b$ means $a \leq Cb$ and $a \sim b$ signifies $a \lesssim b \lesssim a$.

2. The model problem

In this Section we firstly introduce the fractional powers $\mathcal{L}^s$ and the Caffarelli–Silvestre extension, and secondly we describe in detail our proposed discretization which is based on the diagonalization technique introduced in [6].

2.1. Fractional Powers of Elliptic Operators

The power $\mathcal{L}^s$, as in [6, 5], is defined following the spectral theory. Consider the countable collection of eigenpairs $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times H_0^1(\Omega)$, of the problem

$$a_\Omega(\varphi, v) = \lambda(\varphi, v)_{L^2(\Omega)} \ \forall v \in H_0^1(\Omega) \tag{3}$$

where $a_\Omega(\cdot,\cdot)$ is the inner product on $H_0^1(\Omega)$ induced by $\mathcal{L}$ given by

$$a_\Omega(w, v) = \int_\Omega (\nabla w \cdot \nabla v + cwv)\, dx' \tag{4}$$

with real eigenvalues λ_k enumerated in increasing order, counting multiplicities. It is assumed that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $(H_0^1(\Omega), a_\Omega(\cdot,\cdot))$. Then, for $s \geq 0$, we introduce the spaces

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k\varphi_k : \|w\|^2_{\mathbb{H}^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s w_k^2 < \infty \right\}$$

while $\mathbb{H}^{-s}(\Omega)$ denotes the dual space of $\mathbb{H}^s(\Omega)$.

It is known that for functions $w = \sum_k w_k\varphi_k \in \mathbb{H}^1(\Omega)$, the operator $\mathcal{L} : \mathbb{H}^1(\Omega) \to \mathbb{H}^{-1}(\Omega)$ takes the form $\mathcal{L}w = \sum_k \lambda_k w_k \varphi_k$. Then, for $s \in (0, 1)$

and $w = \sum_k w_k \varphi_k \in \mathbb{H}^s(\Omega)$, the operator $\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$ is naturally defined by

$$\mathcal{L}^s w = \sum_{k=1}^{\infty} \lambda_k^s w_k \varphi_k.$$

2.2. The local extended problem

To achieve an effective computational discretization scheme, following [6], we consider a strategy proposed by Caffarelli and Silvestre [2], and subsequently extended by Cabré and Tan [1] and Stinga and Torrea [4] for bounded domains Ω, to localize it. This strategy involves solving the following singular elliptic boundary value problem posed on the extended cylinder $\mathcal{C} = \Omega \times (0, +\infty)$:

$$\begin{aligned} -\text{div}\,(y^\alpha \nabla \mathcal{U}) + \bar{c}(x) y^\alpha \mathcal{U} &= 0 && \text{in } \mathcal{C} \\ \mathcal{U} &= 0 && \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha} \mathcal{U} &= d_s f && \text{on } \Omega \times \{0\} \end{aligned} \tag{5}$$

where $\partial_L \mathcal{C} := \partial\Omega \times (0, \infty)$ is the lateral boundary of $\mathcal{C}$, $d_s := 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)} > 0$ is a normalization constant and $\alpha := 1 - 2s \in (-1, 1)$. The conormal exterior derivative of $\mathcal{U}$ at $\Omega \times \{0\}$ is defined by

$$\partial_{\nu^\alpha} \mathcal{U} = - \lim_{y \to 0+} y^\alpha \partial_y \mathcal{U}. \tag{6}$$

The limit in (6) is understood in the distributional sense [1, 2].

In order to analyze the problem (5) we need to introduce additional spaces. Throughout the text, we denote $x = (x', y) \in \mathcal{C}$ with $x' \in \Omega$ and $y > 0$. If $D \subset \mathbb{R}^n$, we define $L^2(y^\alpha, D)$ as the Lebesgue space for the measure $|y|^\alpha \, dx$. and the weighted Sobolev space

$$H^1(y^\alpha, D) = \left\{ w \in L^2(y^\alpha, D) : \, |\nabla w| \in L^2(y^\alpha, D) \right\}$$

where ∇w is the distributional gradient of w. We equip $H^1(y^\alpha, D)$ with the norm

$$||w||_{H^1(y^\alpha, D)} = \left(||w||^2_{L^2(y^\alpha, D)} + ||\nabla w||^2_{L^2(y^\alpha, D)} \right)^{\frac{1}{2}}. \tag{7}$$

Define the weighted Sobolev space

$$\overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}) : \, w = 0 \, \text{on} \, \partial_L \mathcal{C} \right\} \tag{8}$$

and the bilinear for $a_{\mathcal{C}} : \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}) \times \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}) \to \mathbb{R}$ by

$$a_{\mathcal{C}}(v, w) = \int_{\mathcal{C}} y^\alpha (\nabla v \cdot \nabla w + cvw) \, dx' \, dy. \tag{9}$$

It can be proven as a consequence of a Poincaré's inequality that $a_{\mathcal{C}}(\cdot, \cdot)$ is continuous and coercive. Consequently, it induces an inner product on $\overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$ and the energy norm $\| \cdot \|_{\mathcal{C}}$:

$$\|v\|^2_{\mathcal{C}} := a_{\mathcal{C}}(v, v) \sim \|\nabla v\|^2_{L^2(y^\alpha, \mathcal{C})}. \tag{10}$$

The weak formulation of (5) reads as follows: find $\mathcal{U} \in \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$ such that

$$a_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = d_s \langle f, tr \, \mathcal{V} \rangle \quad \forall \, \mathcal{V} \in \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C}), \tag{11}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $L^2(\Omega)$ and $\operatorname{tr} \mathcal{V}$ is the trace $\mathcal{V}|_{\Omega \times \{0\}}$.

The connection between both problems follows from this fundamental result (see [1, Proposition 2.2] and [4, Theorem 1.1]): given $f \in \mathbb{H}^{-s}(\Omega)$, let $u \in \mathbb{H}^s(\Omega)$ be the solution of (1). If $\mathcal{U} \in \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$ solves (5) then

$$u = \operatorname{tr} \mathcal{U} \qquad \text{and} \qquad d_s \mathcal{L}^s u = \partial_{\nu^\alpha} \mathcal{U} \qquad \text{in } \Omega.$$

2.3. Semi-discretization of the extended problem

Let $\mathcal{Y} > 0$ and $M \in \mathbb{N}$. Given a partition $\mathcal{G}^M$ of $[0, \mathcal{Y}]$ into M subintervals, we will define in this Subsection a semidiscrete approximation $\mathcal{U}_M$ of the solution $\mathcal{U}$ of (11), with $\mathcal{U}_M$ supported on the truncated cylinder $\mathcal{C}_{\mathcal{Y}} := \Omega \times (0, \mathcal{Y})$.

Let $S^1_{\{\mathcal{Y}\}}\left((0, \mathcal{Y}), \mathcal{G}^M\right)$ be the space of piecewise linear functions on $\mathcal{G}^M$ that vanish on $y = \mathcal{Y}$. We consider the space

$$\mathbb{V}_M = H^1_0(\Omega) \otimes S^1_{\{\mathcal{Y}\}}\left((0, \mathcal{Y}), \mathcal{G}^M\right).$$

Functions in $\mathbb{V}_M$ can be extended by 0 to the entire cylinder $\mathcal{C}$ and thus we can consider $\mathbb{V}_M$ as a subspace of $\overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$, that is, $\mathbb{V}_M \subset \overset{\circ}{H}{}^1(y^\alpha, \mathcal{C})$. We can obtain the approximation $\mathcal{U}_M$ as the solution of the semidiscrete problem: find $\mathcal{U}_M \in \mathbb{V}_M$ such that

$$a_{\mathcal{C}}(\mathcal{U}_M, \mathcal{V}) = d_s \langle f, \operatorname{tr} \mathcal{V} \rangle \qquad \forall \mathcal{V} \in \mathbb{V}_M. \tag{12}$$

Let $\{(\mu_i, v_i)\}_{i=1}^M \subset \mathbb{R} \times S^1_{\{\mathcal{Y}\}}\left((0,\mathcal{Y}), \mathcal{G}^M\right) \setminus \{0\}$ be the set of eigenpairs defined by

$$\mu_i \int_0^{\mathcal{Y}} y^\alpha v_i'(y) w'(y)\, dy = \int_0^{\mathcal{Y}} y^\alpha v_i(y) w(y)\, dy \qquad \forall w \in S^1_{\{\mathcal{Y}\}}\left((0,\mathcal{Y}), \mathcal{G}^M\right)$$

with $\{v_i\}$ normalized such that

$$\int_0^{\mathcal{Y}} y^\alpha v_i'(y) v_j'(y)\, dy = \delta_{ij}, \qquad \int_0^{\mathcal{Y}} y^\alpha v_i(y) v_j(y)\, dy = \mu_i \delta_{ij}.$$

Then it can be easily verified that we can write

$$\mathcal{U}_M(x', y) = \sum_{i=1}^M U_i(x') v_i(y) \tag{13}$$

with $U_i \in H_0^1(\Omega)$, $i = 1, \ldots, M$ being the solutions of the problems

$$\mu_i \left(\nabla U_i, \nabla V\right) + ((1 + \bar{c}(x)) U_i, V) = d_s v_i(0) \langle f, V \rangle \qquad \forall V \in H_0^1(\Omega). \tag{14}$$

Problems (14) are of reaction–diffusion type and they become singularly perturbed when the eigenvalues μ_i are small. In order to obtain a linear (in M) approximation $\mathcal{U}_M$ of $\mathcal{U}$ we will later consider partitions $\mathcal{G}^M$ which, in particular, have a very small first interval. In such cases, small eigenvalues μ_i occur. According to [6, Lemma 18] we have the upper bound

$$\mu_i \leq \mathcal{Y}^2 (1 - \alpha^2)^{-1}, \qquad i = 1, \ldots, M. \tag{15}$$

On the other hand, following [6, Lemma 19] it is possible to prove the lower bound

$$\mu_i \gtrsim (\log M)^{-\alpha\sigma}\, l_{min}^2, \qquad i = 1, \ldots, M \tag{16}$$

where l_{min} denotes the minimal length of the intervals in $\mathcal{G}^M$ and the constant involved depends only on s. See Remark 2.3 later on for a proof. Thus we see that a number of the reaction–diffusion problems (14) become singularly perturbed. Therefore, some care has to be taken in the approximation of the functions U_i.

2.4. The full discretization

In this Subsection we introduce our finite element approximation based on the semi-discretization previously introduced. Now we restrict ourselves to the case $\Omega = (0,1)^2$. Let σ be a parameter satisfying

$$1 - s < \sigma < 1. \tag{17}$$

Let $M \in \mathbb{N}$ and $\mathcal{Y} = c \log M$ with the constant c to be chosen later. We then define the partition $\mathcal{G}^M = \{I_i\}_{i=1}^M$ in $[0, \mathcal{Y}]$, where $I_i = [y_{i-1}, y_i]$ with

$$y_i = \mathcal{Y} \left(\frac{i}{M} \right)^{\frac{1}{1-\sigma}}, \quad i = 0, \ldots, M. \tag{18}$$

A discrete approximation $\mathcal{U}_{M,N}$ of $\mathcal{U}_M$ defined in (13) is constructed by means of finite element discretizations of the problems (14). Let $N \in \mathbb{N}$. We introduce a graded mesh $\mathcal{T}_N$ of Ω obtained as a tensor product of partitions of the interval $[0,1]$ into $2N$ subintervals. Given a parameter η satisfying

$$\frac{3}{4} \leq \eta < 1, \tag{19}$$

let $\xi_0, \xi_1, \ldots, \xi_N$ be the grid points on the interval $[0, \frac{1}{2}]$ given by

$$\xi_i = \frac{1}{2} \left(\frac{i}{N} \right)^{\frac{1}{1-\eta}}, \quad i = 0, \ldots, N. \tag{20}$$

This partition is extended to a grid $\{\xi_0, \xi_1, \ldots, \xi_N, \ldots, \xi_{2N}\}$ of $[0,1]$ by setting $\xi_i = 1 - \xi_{2N-i}$ for $i = N+1, \ldots, 2N$. For $1 \leq i, j \leq 2N$ let $R_{ij} = [\xi_{i-1}, \xi_i] \times [\xi_{j-1}, \xi_j]$. Then we obtain a graded mesh $\mathcal{T}_N = \{R_{ij}\}_{i,j=1}^{2N}$ of $\overline{\Omega}$. Additionally, we set $h_i = \xi_i - \xi_{i-1}$.

Remark 2.1. *The intervals $I_i, i = 2, \ldots, M$ of the partition $\mathcal{G}^M$, satisfy*

$$|I_i| \leq C \mathcal{Y} \frac{1}{M} y^\sigma = C (\log M) \frac{1}{M} y^\sigma, \qquad \forall y \in I_i, \tag{21}$$

with a constant C depending only on s.

Indeed, with $i \geq 2$, for some $\zeta \in (i-1, i)$ we have

$$\begin{aligned} y_i - y_{i-1} &= \left[\left(\frac{i}{M} \right)^{\frac{1}{1-\sigma}} - \left(\frac{i-1}{M} \right)^{\frac{1}{1-\sigma}} \right] \mathcal{Y} \\ &= \frac{1}{1-\sigma} \left(\frac{\zeta}{M} \right)^{\frac{\sigma}{1-\sigma}} \frac{1}{M} \mathcal{Y}. \end{aligned}$$

But

$$\left(\frac{\zeta}{M}\right)^{\frac{\sigma}{1-\sigma}} = \left(\frac{i-1}{M}\right)^{\sigma} \left(\frac{\zeta}{i-1}\right)^{\frac{\sigma}{1-\sigma}} \left(\frac{M}{i-1}\right)^{-\frac{\sigma^2}{1-\sigma}}.$$

Then, (21) *follows from* $\mathcal{Y} = c \log M$*,*

$$\left(\frac{\zeta}{i-1}\right)^{\frac{\sigma}{1-\sigma}} \leq 2^{\frac{\sigma}{1-\sigma}}, \qquad \left(\frac{M}{i-1}\right)^{-\frac{\sigma^2}{1-\sigma}} \leq 1$$

and

$$\left(\frac{i-1}{M}\right)^{\sigma} \leq y^{\sigma} \quad \forall y \in I_i.$$

Remark 2.2. *For* $i \geq 2$ *we also have*

$$y \leq C (\log M)^{\sigma} z \qquad \forall y, z \in I_i. \tag{22}$$

Indeed, we have from (21) *that*

$$y_i \leq y_{i-1} + C (\log M) \frac{1}{M} y_{i-1}^{\sigma}.$$

Since $y_{i-1} \geq y_1 = \mathcal{Y} \left(\frac{1}{M}\right)^{\frac{1}{1-\sigma}}$*, and then* $y_{i-1}^{\sigma-1} \leq c(logM)^{\sigma-1}M$*, we obtain*

$$y_i \leq y_{i-1} (1 + C (\log M)^{\sigma})$$

which implies (22)*.*

Associated with $\mathcal{T}_N$, we introduce the standard piecewise bilinear finite element space

$$V_N = \{v \in H_0^1(\Omega) : v|_{R_{ij}} \in \mathcal{Q}_1(R_{ij}), 1 \leq i, j \leq 2N\}, \tag{23}$$

where $\mathcal{Q}_1(R)$ denotes the space of bilinear functions on the rectangle R. Now we can define $U_{i,N}$, $i = 1, \ldots, M$ as the solutions of problems: find $U_{i,N} \in V_N$ such that

$$\mu_i (\nabla U_{i,N}, \nabla V) + ((1 + \bar{c}(x))U_{i,N}, V) = d_s v_i(0)\langle f, V\rangle \qquad \forall V \in V_N. \tag{24}$$

For each $U_{i,N}$ we will define in Section 4 a post-processed $U_{i,N}^*$ with improved approximation properties. Then, similar to (13) we define

$$\mathcal{U}_{M,N}(x', y) = \sum_{i=1}^{M} U_{i,N}^*(x') v_i(y), \tag{25}$$

and finally, the approximation of u is given by

$$u_{M,N}(x',y) = \mathrm{tr}\,\mathcal{U}_{M,N}(x',y) = \sum_{i=1}^{M} U^*_{i,N}(x')v_i(0). \tag{26}$$

Remark 2.3. *With the definitions introduced in this Subsection, we can prove* (16)*. Using a standard rescaling argument $y = |I_1|\hat{y}$ to map intervals $[0,1]$ onto I_1, and the equivalence of norms for linear functions on I_1, we have for the eigenfunctions v_i (defined in Subsection 2.3)*

$$\|v_i'\|_{L^2(y^\alpha,I_1)} \sim |I_1|^{-1}\|v_i\|_{L^2(y^\alpha,I_1)}.$$

Using another scaling argument and the equivalence (22) *on interval I_i for $i \geq 2$, we have*

$$\|v_i'\|_{L^2(y^\alpha,I_i)} \sim (\log M)^{\frac{\alpha\sigma}{2}}\,|I_i|^{-1}\|v_i\|_{L^2(y^\alpha,I_i)}.$$

Therefore, by squaring and adding the previous inequalities, we obtain

$$\mu_i = \|v_i\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} \geq C\,(\log M)^{-\alpha\sigma}\,(\min|I_i|)^2\,\|v_i'\|^2_{L^2(y^\alpha,I_i)}$$

and taking into account that $\|v_i'\|_{L^2(y^\alpha,I_i)} = 1$ and $\min|I_i| = |I_1|$ we have

$$\mu_i = \|v_i\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} \geq C\,(\log M)^{-\alpha\sigma}\,|I_1|^2$$

which proves (16)*.*

Remark 2.4. *The factor $v_i(0)$ which appears in the right hand side of problem* (14) *and its discretization* (24) *can be bounded following [6, Lemma 17]. Taking into acount that $v(\mathcal{Y}) = 0$ and $\|v_i'\|_{L^2(y^\alpha,(0,\mathcal{Y}))} = 1$ we have*

$$|v_i(0)| = \left|\int_0^{\mathcal{Y}} v'(y)\,dy\right| = \left|\int_0^{\mathcal{Y}} y^{-\frac{\alpha}{2}}y^{\frac{\alpha}{2}}v'(y)\,dy\right| \leq \frac{\mathcal{Y}^{\frac{1-\alpha}{2}}}{(1-\alpha)^{\frac{1}{2}}} \leq C\,(\log M)^{\frac{1-\alpha}{2}}$$

with the constant C depending on s and independent of M.

Remark 2.5. *Similarly to Remarks 2.1 and 2.2, for all $R_{ij} \in \mathcal{T}_N$, we can prove that*

$$h_i \leq C\frac{1}{N}x_1^\eta, \qquad h_j \leq C\frac{1}{N}x_2^\eta, \qquad \forall (x_1, x_2) \in R_{ij} \tag{27}$$

and

$$x_1 \leq Cw_1, \qquad x_2 \leq Cw_2, \qquad \forall (x_1, x_2), (w_1, w_2) \in R_{ij}. \tag{28}$$

On the other hand it is easy to check that

$$\frac{h_i}{h_{i+1}} \leq C, \quad \forall i = 0, \ldots, 2N-1. \tag{29}$$

Here C is a constant depending only on η.

2.5. Some preliminaries for the error analysis

The error estimate starts with the trace inequality (see [6, Subsection 2.2])

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C_{tr}\|\mathcal{U} - \mathcal{U}_{M,N}\|_{\mathcal{C}},$$

and then by the triangle inequality we have

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C_{tr}\left(\|\mathcal{U} - \mathcal{U}_M\|_{\mathcal{C}} + \|\mathcal{U}_M - \mathcal{U}_{M,N}\|_{\mathcal{C}}\right). \tag{30}$$

In Section 3 we will obtain the estimate

$$\|\mathcal{U} - \mathcal{U}_M\|_{\mathcal{C}} \leq C\left(\log M\right)^m \frac{1}{M}\|f\|_0$$

with an exponent m to be defined later.

On the other hand, since

$$\mathcal{U}_M(x', y) - \mathcal{U}_{M,N}(x', y) = \sum_{i=1}^{M}\left(U_i(x', y) - U_{i,N}^*(x', y)\right)v_i(y)$$

we have, following [6, eq. (6.5)], that

$$\|\mathcal{U}_M - \mathcal{U}_{M,N}\|_{\mathcal{C}}^2 = \sum_{i=1}^{M}\|U_i - U_{i,N}^*\|_{\mu_i,\Omega}^2, \tag{31}$$

where the norm $\|\cdot\|_{\mu_i,\Omega}$ is the energy norm associated with problem (24):

$$\|v\|^2_{\mu_i,\Omega} = \|v\|^2_{\mu_i} := \mu_i \|\nabla v\|^2_0 + \|(1+\bar{c}(x))^{\frac{1}{2}} v\|^2_0.$$

We observe that in order to obtain a linear order of convergence for our discretization, we require superlinear estimates for the approximations of the (singularly perturbed) reaction–diffusion problems defining U_i. In Section 4, we will see that this can be achieved if the parameter η, defining the graded meshes, satisfies (19).

3. Error estimate in the extended domain

In this Section, we estimate the semi-discretization error $\|\mathcal{U} - \mathcal{U}_M\|_{\mathcal{C}}$. In order to do that we need to define an interpolation operator for functions in $C^2([0,\mathcal{Y}], L^2(\Omega))$.

Given a Sobolev space X, following [6], we consider a piecewise linear interpolation operator $\pi^1_{y,\{\mathcal{Y}\}}$ defined over a grid $\mathcal{G}^M$ on $[0,\mathcal{Y}]$ for functions $v \in \mathcal{C}^2([0,\mathcal{Y}], X)$. On the interval I_1, $\pi^1_{y,\{\mathcal{Y}\}} v$ is defined by interpolating v at points $y_1/2$ and y_1; on intervals I_i with $1 < i < M$ the interpolation is at points y_{i-1} and y_i; and finally on I_M it interpolates at y_{M-1} and is enforced to vanish at y_M, $\left(\pi^1_{y,\{\mathcal{Y}\}} v\right)(y_M) = 0$.

Let

$$\omega_{\theta,\gamma}(y) = y^\theta e^{\gamma y}.$$

We will write y^α and $\omega_{\alpha,0}(y)$ interchangeably. For a function $v \in L^2(I, X)$, where X is a Hilbert space and I is a real interval, we introduce the notation

$$\|v\|_{L^2(\omega_{\theta,\gamma},I;X)} = \left(\int_I \omega_{\theta,\gamma}(y) \|v(y)\|^2_X \, dy \right)^{\frac{1}{2}}.$$

When there is no confusion, we will omit the space X writing

$$\|v\|_{L^2(\omega_{\theta,\gamma},I)} = \|v\|_{L^2(\omega_{\theta,\gamma},I;X)}.$$

For a function $\mathcal{V}(x', y)$, with $x' \in \Omega$ and $y \in I$, such that for each y it holds $\mathcal{V}(\cdot, y) \in X$, we write

$$\|\mathcal{V}\|_{L^2(\omega_{\theta,\gamma},\Omega\times I)} := \left(\int_I \omega_{\theta,\gamma}(y) \|\mathcal{V}(\cdot, y)\|^2_X \, dy \right)^{\frac{1}{2}}.$$

We need interpolation error estimates for $\pi^1_{y,\{\mathcal{Y}\}}$. More precisely, we have the classical local estimates

$$\begin{aligned} \left\|v - \pi^1_{y,\{\mathcal{Y}\}} v\right\|_{L^2(\omega_{0,0},I_i)} &\le C|I_i|\,\|v'\|_{L^2(\omega_{0,0},I_i)} \\ \left\|(v - \pi^1_{y,\{\mathcal{Y}\}} v)'\right\|_{L^2(\omega_{0,0},I_i)} &\le C|I_i|\,\|v''\|_{L^2(\omega_{0,0},I_i)} \end{aligned} \tag{32}$$

for each interval I_i, for functions $v \in H^1(I,X)$ and $v \in H^2(I,X)$, respectively. On the interval I_1, we will use the weighted error estimates

$$\begin{aligned} \left\|v - \pi^1_{y,\{\mathcal{Y}\}} v\right\|_{L^2(\omega_{\alpha,0},I_1)} &\le C|I_1|^\beta\,\|v'\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)} \\ \left\|(v - \pi^1_{y,\{\mathcal{Y}\}} v)'\right\|_{L^2(\omega_{\alpha,0},I_1)} &\le C|I_i|^\beta\,\|v''\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)} \end{aligned} \tag{33}$$

which are proven in [6, eqs. (A.6) and (A.4)].

In view of [6, eq. (6.10)], $\mathcal{U}$ can be seen as a function in $C^2([0,\mathcal{Y}], L^2(\Omega)) \cap C^2([0,\mathcal{Y}], H^1_0(\Omega))$ and thus it makes sense to consider $\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}$.

In the proof of the next result, we will use the following estimates for $\mathcal{U}$ taken from [6, Theorem 1]. Let γ be a fixed positive parameter satisfying $\gamma < 2\sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the problem (3). We have

$$\begin{aligned} \|\partial_y \mathcal{U}\|_{L^2(\omega_{\alpha-2\tilde\nu,\gamma},\mathcal{C})} &\le C\|f\|_{\mathbb{H}^{-s+\tilde\nu}(\Omega)} \\ \|\partial^2_y \mathcal{U}\|_{L^2(\omega_{\alpha+2-2\tilde\nu,\gamma},\mathcal{C})} &\le C\|f\|_{\mathbb{H}^{-s+\tilde\nu}(\Omega)} \\ \|\partial_y \nabla_{x'} \mathcal{U}\|_{L^2(\omega_{\alpha+2-2\nu,\gamma},\mathcal{C})} &\le C\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)} \end{aligned} \tag{34}$$

with $0 \le \tilde\nu < s = (1-\alpha)/2$ and $0 \le \nu < 1+s$.

Proposition 3.1. *Assume $f \in L^2(\Omega)$. We consider the approximation $\mathcal{U}_M$ defined by (12), supported on the truncated cylinder $\mathcal{C}_\mathcal{Y}$, obtained using the grid $\mathcal{G}^M$ of $[0,\mathcal{Y}]$ defined by (18), where $\mathcal{Y} = c\log M$ with $c > \frac{3}{\gamma}$. Then it follows that*

$$\|\mathcal{U} - \mathcal{U}_M\|_\mathcal{C} \le C(\log M)^{\frac{3+\alpha\sigma}{2}} \frac{1}{M}\|f\|_0, \tag{35}$$

where C is a constant depending only on s.

Proof. From the Galerkin orthogonality we have

$$\|\mathcal{U} - \mathcal{U}_M\|_\mathcal{C} \le \left\|\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right\|_\mathcal{C}. \tag{36}$$

It follows from the Poincaré's inequality [6, ineq. (2.7)] for functions in $\overset{\circ}{H^1}(y^\alpha,\Omega)$ that the seminorm $\|\nabla(\cdot)\|_{L^2(y^\alpha,\mathcal{C})}$ is equivalent to the norm $\|\cdot\|_\mathcal{C}$. Then

$$\|\mathcal{U} - \mathcal{U}_M\|_\mathcal{C} \lesssim \left\|\nabla\left(\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right)\right\|_{L^2(y^\alpha,\mathcal{C})}$$

Since $\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}$ vanishes outside $\mathcal{C}_\mathcal{Y}$ we have

$$\left\|\nabla\left(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right)\right\|_{L^2(y^\alpha,\mathcal{C})} \leq \left\|\nabla\left(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right)\right\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}. \tag{37}$$

From [6, eq. (5.8)] we have that the second term on the right hand side of (37) is exponentially small in $\mathcal{Y}$, in fact,

$$\|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})} \lesssim e^{-\gamma\mathcal{Y}/2}\|f\|_{H^{-s}(\Omega)}.$$

Taking into account that

$$\mathcal{Y} = c\log M$$

with $c \geq 2/\gamma$ we obtain

$$\|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})} \lesssim \frac{1}{M}\|f\|_{H^{-s}(\Omega)}. \tag{38}$$

Now we consider the first term of the right hand side of (37). We have

$$\begin{aligned}\|\nabla(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} &\\ \leq \|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} &+ \|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})}\\ &=: A+B,\end{aligned} \tag{39}$$

where we have used that $\nabla_{x'}\left(\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U}\right) = \pi^1_{y,\{\mathcal{Y}\}}\left(\nabla_{x'}\mathcal{U}\right)$.

We can estimate A as follows. On I_1, using the first inequality of (33) with $v = \nabla_{x'}\mathcal{U}$ we have

$$\|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,I_1)} \leq C|I_1|^\beta\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)}$$

for $\beta \geq 0$. Taking $\beta = 1-\sigma$, and since

$$|I_1| = c\log(M)\left(\frac{1}{M}\right)^{\frac{1}{1-\sigma}},$$

we obtain

$$\|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,I_1)} \leq C(\log M)^{1-\sigma}\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_1)}. \tag{40}$$

On intervals $I_i, i = 2,\ldots,M-1$, using the first inequality of (32) we have

$$\|\nabla_{x'}\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{0,I_i} \leq C|I_i|\|\partial_y\nabla_{x'}\mathcal{U}\|_{0,I_i},$$

and, taking into account (21) and (22), we obtain

$$\|\nabla_{x'}\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\|_{L^2(y^\alpha,I_i)} \leq C\left(\log M\right)^{1+\frac{\sigma\alpha}{2}}\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_i)}.$$

Finally, for I_M, if π^1_y is the interpolation operator on the grid $\mathcal{G}^M$ defined like $\pi^1_{y,\{\mathcal{Y}\}}$ but without imposing $\pi^1_y(\cdot)(y_M) = 0$ (that is, $\pi^1_y(v)(y_M) = v(y_M)$), we have

$$\begin{aligned}\left\|\nabla_{x'}\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \leq& \left\|\nabla_{x'}\mathcal{U} - \pi^1_y\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)}\\ &+ \left\|\left(\pi^1_y - \pi^1_{y,\{\mathcal{Y}\}}\right)\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)}\end{aligned} \tag{41}$$

The first term can be bounded in the same way as for $I_i, 2 \leq i \leq M-1$, obtaining

$$\left\|\nabla_{x'}\mathcal{U} - \pi^1_y\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \leq C\left(\log M\right)^{1+\frac{\sigma\alpha}{2}}\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_M)},$$

and for the second one we have

$$\left\|\left(\pi^1_y - \pi^1_{y,\{\mathcal{Y}\}}\right)\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \leq C\left(\log M\right)^{\frac{\alpha}{2}}|I_M|^{\frac{1}{2}}\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_{L^2(\Omega)},$$

since $\pi^1_y - \pi^1_{y,\{\mathcal{Y}\}}$ is a linear function in the variable y with values in $L^2(\Omega)$ vanishing at $y = y_{M-1}$. Inserting the previous inequalities into (41) we obtain

$$\begin{aligned}&\left\|\nabla_{x'}\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\nabla_{x'}\mathcal{U}\right\|_{L^2(y^\alpha,I_M)} \leq\\ &\qquad C\left(\log M\right)^{1+\frac{\sigma\alpha}{2}}\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_M)}+\\ &\qquad\qquad\left(\log M\right)^{\frac{\alpha}{2}}|I_M|^{\frac{1}{2}}\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_0.\end{aligned} \tag{42}$$

Using [6, eq. (A.10) and Lemma 16], we have

$$\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_0 \leq \mathcal{Y}^{-\frac{\alpha}{2}-1+\beta}e^{-\mathcal{Y}\gamma/2}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,\gamma},\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}$$

and since $|I_M| \leq C(\log M)\frac{1}{M}\mathcal{Y}^\sigma$ it results

$$\begin{aligned}&\left(\log M\right)^{\frac{\alpha}{2}}|I_M|^{\frac{1}{2}}\|\nabla_{x'}\mathcal{U}(\cdot,\mathcal{Y})\|_0 \leq\\ &\quad C\left(\log M\right)^{\frac{1+\alpha}{2}}\left(\frac{1}{M}\right)^{\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-1+\beta+\frac{\sigma}{2}}e^{-\mathcal{Y}\gamma/2}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,\gamma},\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}.\end{aligned} \tag{43}$$

Taking again $\beta = 1-\sigma$ and since

$$\max\left\{1-\sigma,\, 1+\frac{\sigma\alpha}{2}, \frac{1+\alpha}{2}\right\} = 1+\frac{\sigma\alpha}{2}$$

we have from inequalities (40)-(43) that

$$A \leq C(\log M)^{1+\frac{\sigma\alpha}{2}}\left[\frac{1}{M}\|\partial_y \nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} + \left(\frac{1}{M}\right)^{\frac{1}{2}} \mathcal{Y}^{-\frac{\alpha+\sigma}{2}} e^{-\mathcal{Y}\gamma/2}\|\partial_y \nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})}\right]. \quad (44)$$

It remains to estimate the term B in equation (39). Using the second interpolation error estimate from (33) it follows

$$\begin{aligned}\|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} &\leq |I_1|^\beta \|\partial^2_y \mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,0},I_1)} \\ &\leq C(\log M)^{1-\sigma}\tfrac{1}{M}\|\partial^2_y\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_1)}\end{aligned} \quad (45)$$

if $\beta = 1-\sigma$. On intervals $I_i, i = 2,\ldots,M-1$ using again the standard error estimates (32) and properties (21)–(22) of $\mathcal{G}^M$, we obtain

$$\|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,I_i)} \leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial^2_y\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_i)}. \quad (46)$$

For the interval I_M we have again (recall the definition of $\pi^1_{\mathcal{Y}}$ before equation (41))

$$\begin{aligned}\|\partial_y(\mathcal{U}-\pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,I_M)} &\leq \|\partial_y(\mathcal{U}-\pi^1_y\mathcal{U})\|_{L^2(y^\alpha,I_M)} + \\ &\qquad \left\|\partial_y\left[\left(\pi^1_y-\pi^1_{y,\{\mathcal{Y}\}}\right)\mathcal{U}\right]\right\|_{L^2(y^\alpha,I_M)} \\ &\leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial^2_y\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},I_M)} + \\ &\qquad C\,(\log M)^{\frac{\alpha}{2}}\,|I_M|^{-\frac{1}{2}}\|\mathcal{U}(\cdot,\mathcal{Y})\|_{0,\Omega}.\end{aligned}$$

Then, since $|I_M| \gtrsim \frac{1}{M}$, and taking again [6, eq. (A.10) and Lemma 16] into account, it follows

$$(\log M)^{\frac{\alpha}{2}}\,|I_M|^{-\frac{1}{2}}\|\mathcal{U}(\cdot,\mathcal{Y})\|_0 \leq (\log M)^{\frac{\alpha}{2}}\left(\frac{1}{M}\right)^{-\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-1+\beta}e^{-\mathcal{Y}\gamma/2}\|\partial_y\mathcal{U}\|_{L^2(\omega_{\alpha+2-2\beta,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})}. \quad (47)$$

Hence, from inequalities (45)–(47) with $\beta = 1 - \sigma$ we obtain

$$B \leq C\,(\log M)^{1+\frac{\sigma\alpha}{2}}\,\frac{1}{M}\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})}+ \\ C\,(\log M)^{\frac{\alpha}{2}}\left(\frac{1}{M}\right)^{-\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-\sigma}e^{-\mathcal{Y}\gamma/2}\|\partial_y\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})}. \quad (48)$$

Inserting (44) and (48) into (39) we have

$$\|\nabla(\mathcal{U}-\pi^1_{y,\mathcal{Y}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \leq \\ C(\log M)^{1+\frac{\alpha\sigma}{2}}\Bigg[\frac{1}{M}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})}+\frac{1}{M}\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})}+ \\ \left(\frac{1}{M}\right)^{\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha+\sigma}{2}}e^{-\mathcal{Y}\gamma/2}\|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})}+ \\ \left(\frac{1}{M}\right)^{-\frac{1}{2}}\mathcal{Y}^{-\frac{\alpha}{2}-\sigma}e^{-\mathcal{Y}\gamma/2}\|\partial_y\mathcal{U}\|_{\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}}}\Bigg] \quad (49)$$

Since we take σ verifying (17), that is

$$\frac{\alpha+1}{2} < \sigma < 1$$

then we have from (34) that

$$\|\partial_y^2\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} \lesssim \|f\|_{-s+1-\sigma} \leq \|f\|_0, \\ \|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} \lesssim \|f\|_{-s+1-\sigma} \leq \|f\|_0, \\ \|\partial_y\nabla_{x'}\mathcal{U}\|_{L^2(\omega_{\alpha+2\sigma,0},\mathcal{C}_{\mathcal{Y}})} \lesssim \|f\|_{-s+1-\sigma} \leq \|f\|_0,$$

and since for a fixed $\gamma_0 > 0$ it holds $y^{2\sigma} \leq Ce^{\gamma_0 y}$ for all $y \geq 1$ we also have

$$\|\partial_y\mathcal{U}\|^2_{L^2(\omega_{\alpha+2\sigma,\gamma},\mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} = \int_{\mathcal{Y}}^\infty \|\partial_y\mathcal{U}\|_0^2\,y^{\alpha+2\sigma}e^{\gamma y}\,dy \\ \lesssim \int_{\mathcal{Y}}^\infty \|\partial_y\mathcal{U}\|_0^2\,y^{\alpha}e^{(\gamma+\gamma_0)y}\,dy \lesssim \|f\|^2_{-s} \leq \|f\|_0^2$$

if γ_0 is taken such that $0 \leq \gamma + \gamma_0 < 2\sqrt{\lambda_1}$. Then from (49) we have

$$\|\nabla(\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \leq C(\log M)^{1+\frac{\alpha\sigma}{2}} \left\{ \frac{1}{M} + \left(\frac{1}{M}\right)^{\frac{1}{2}} \mathcal{Y}^{-\frac{\alpha+\sigma}{2}} e^{-\mathcal{Y}\gamma/2} + \left(\frac{1}{M}\right)^{-\frac{1}{2}} \mathcal{Y}^{-\frac{\alpha}{2}-\sigma} e^{-\mathcal{Y}\gamma/2} \right\} \|f\|_0.$$

Now, we need to consider that

$$\mathcal{Y} = c \log M$$

with $c > \frac{3}{\gamma}$ in order to obtain

$$\|\nabla(\mathcal{U} - \pi^1_{y,\{\mathcal{Y}\}}\mathcal{U})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \leq C(\log M)^{\frac{3+\alpha\sigma}{2}} \frac{1}{M} \|f\|_0. \tag{50}$$

Inequality (37), together with (38) and (50), give the result. □

4. Superconvergent approximations of a reaction–diffusion equation using graded meshes

The goal of this Section is to prove superconvergence results for the standard $\mathcal{Q}_1$ finite element approximation of the reaction–diffusion model problem introduced below when appropriated graded meshes are used.

We consider the model problem

$$\begin{aligned} -\varepsilon^2 \Delta w + b(x) w &= f \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{51}$$

where $\Omega = (0,1)^2$, $0 < \varepsilon \ll 1$ is a small positive parameter and

$$b(x_1, x_2) \geq 1 \text{ in } \Omega. \tag{52}$$

4.1. Auxiliary results

We will assume that $f \in \mathcal{C}^2([0,1]^2)$ and that it satisfies the compatibility conditions

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0. \tag{53}$$

It is known that under these hypotheses, the exact solution of problem (51) satisfies $w \in \mathcal{C}^4(\Omega) \cap \mathcal{C}^2(\overline{\Omega})$. Moreover, we have the following pointwise estimates for w and its derivatives (see [14, Lemma 4.1]): if $0 \le k \le 4$ then

$$\left| \frac{\partial^k w}{\partial x_1^k}(x_1, x_2) \right| \le C \left(1 + \varepsilon^{-k} e^{-x_1/\varepsilon} + \varepsilon^{-k} e^{-(1-x_1)/\varepsilon} \right), \tag{54}$$

$$\left| \frac{\partial^k w}{\partial x_2^k}(x_1, x_2) \right| \le C \left(1 + \varepsilon^{-k} e^{-x_2/\varepsilon} + \varepsilon^{-k} e^{-(1-x_2)/\varepsilon} \right). \tag{55}$$

We also have some weighted a priori estimates for w which are uniform in the perturbation parameter ε (see [8, Lemma 3.1]): let $d(t) = \min\{t, 1-t\}$ be the distance to the boundary function on the interval $[0, 1]$, then

(i) if $0 \le k \le 4$, $\alpha + \beta \ge k - \frac{1}{2}, \alpha \ge 0, \beta > -\frac{1}{2}$ then

$$\varepsilon^\alpha \left\| d(x_1)^\beta \frac{\partial^k w}{\partial x_1^k} \right\|_0 \le C, \quad \varepsilon^\alpha \left\| d(x_2)^\beta \frac{\partial^k w}{\partial x_2^k} \right\|_0 \le C, \tag{56}$$

(ii) if $\alpha + \beta \ge \frac{5}{2}, \alpha \ge \frac{3}{4}, \beta > \frac{1}{2}$ then

$$\varepsilon^\alpha \left\| d(x_2)^\beta \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_0 \le C. \tag{57}$$

4.2. Finite element approximation on graded meshes

The standard weak formulation of problem (51) is: find $w \in H_0^1(\Omega)$ such that

$$\mathcal{B}(w, v) = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega),$$

where the bilinear form $\mathcal{B}$ is defined as

$$\mathcal{B}(w, v) = \int_\Omega (\varepsilon^2 \nabla w \cdot \nabla v + bwv) \, dx.$$

For a domain D, we will work with the ε-weighted H^1-norm (refereed as ε-norm in what follows) defined by

$$\|v\|_{\varepsilon^2, D}^2 = \varepsilon^2 \|\nabla v\|_{0,D}^2 + \|v\|_{0,D}^2.$$

When $D = \Omega$, for simplicity, we drop the subscript Ω.

It is well known that under the hypothesis (52), the bilinear form $\mathcal{B}$ is uniformly continuous and coercive in the ε-norm, in particular

$$\|v\|_{\varepsilon^2}^2 \leq \mathcal{B}(v,v) \quad \forall v \in H_0^1(\Omega).$$

In [7], an analysis for the approximation of problem (51) by bilinear finite elements using appropriate graded meshes was developed. Almost optimal convergence, uniform with respect to ε, was proven in that paper. The graded meshes used in [7], which depend on a parameter η, with $\frac{1}{2} < \eta < 1$, are constructed independently of the perturbation parameter ε. In [8], under the stronger restriction $\frac{3}{4} \leq \eta < 1$, supercloseness results for the same scheme considered in [7] were obtained. Specifically, the difference between the finite element solution and the Lagrange interpolant of the exact solution, in the ε-norm, is of higher order than the error itself. The constants in such estimates depend only weakly on the singular perturbation parameter. In this Section our aim is, starting from these known results, to obtain a higher order approximation by a local post-processing of the computed solution.

On $\Omega = (0,1)^2$, for $N \in \mathbb{N}$, $h = 1/N$ and a given grading parameter η we consider the mesh $\mathcal{T}_N$ introduced in Subsection 2.4. Associated with $\mathcal{T}_N$, we introduce the piecewise bilinear finite element space V_N defined by (23), and the finite element approximation $w_N \in V_N$ that solves

$$\mathcal{B}(w_N, v) = \int_\Omega f v \, dx \quad \forall v \in V_N.$$

4.3. A higher order approximation by post-processing

As it is known, the supercloseness estimate (see [8, Theorem 4.7]):

$$\|w_N - w_I\|_{\varepsilon^2} \leq C h^2 \log^{\frac{1}{2}}(\tfrac{1}{\varepsilon}), \tag{58}$$

where $w_I \in V_N$ is the Lagrange interpolant of the exact solution w, can be used to improve the numerical approximation by a local post-processing.

Remark 4.1. *We note that the graded meshes over Ω used in [8] have been defined differently than in Subsection 2.4. However, the only properties of the mesh involved in the proof of inequality* (58) *are those given in Remark 2.5. Therefore, the inequality remains valid even with our definition of the meshes.*

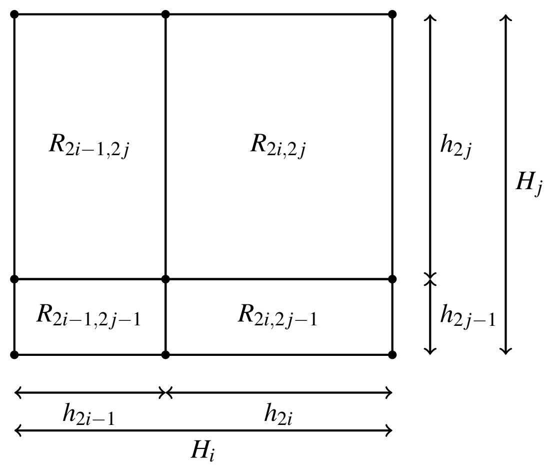

Figure 1: Element S_{ij}

We will define the post-processed w_N^* of the finite element solution w_N following [11, 15, 10]. We repeat the construction given in those papers for the sake of completeness. Since $\mathcal{T}_N$ is a tensor product mesh of a partition of $[0,1]$ with $2N$ subintervals, it can be viewed as a refinement of the coarser mesh $\mathcal{S}_N$, formed by elements S_{ij} with $1 \leq i, j \leq N$ as described in Figure 1. Let I_2 be the biquadratic interpolation operator over the mesh $\mathcal{S}_N$, which for a function $v \in \mathcal{C}(\Omega)$, is defined on each S_{ij} as the Lagrange interpolant over the nine nodes indicated in Figure 1, i.e., the vertices of the four elements of the finer mesh $\mathcal{T}_N$ contained within S_{ij}. Consider

$$w_N^* := I_2 w_N.$$

Then we want to show that w_N^* is a second order approximation of w in the ε-norm.

We will need the following estimates for the operator I_2.

Lemma 4.1. *There exists a constant C such that, for any $v \in V_N$ and $S_{ij} \in \mathcal{S}_N$ we have*

$$\|I_2 v\|_{L^\infty(S_{ij})} \leq C \|v\|_{L^\infty(S_{ij})}. \tag{59}$$

Proof. For $(x_1, x_2) \in S_{ij}$ and $\alpha, \beta \in \{1, 2, 3\}$ fixed, we define

$$\varphi_{\alpha\beta}(x_1, x_2) = \prod_{k \neq \alpha} \frac{x_1 - x_1^k}{x_1^\alpha - x_1^k} \prod_{\ell \neq \beta} \frac{x_2 - x_2^\ell}{x_2^\beta - x_2^\ell} \tag{60}$$

where (x_1^k, x_2^ℓ), with $k, \ell = 1, 2, 3$, are the interpolation nodes on S_{ij}. Then, we can write

$$I_2 v = \sum_{\alpha,\beta=1,2,3} v(x_1^\alpha, x_2^\beta)\, \varphi_{\alpha\beta}, \qquad \text{on } S_{ij}. \tag{61}$$

Setting H_i and H_j as the lengths of the element S_{ij} along the directions of the x_1 and x_2 axes respectively, as in Figure 1, and $h_{min}^{x_1} := \min\{h_{2i-1}, h_{2i}\}$, $h_{min}^{x_2} := \min\{h_{2j-1}, h_{2j}\}$, we have that $|x_1 - x_1^k| \le H_i$, $|x_2 - x_2^\ell| \le H_j$, $|x_1^\alpha - x_1^k| \ge h_{min}^{x_1}$ and $|x_2^\beta - x_2^\ell| \ge h_{min}^{x_2}$.

Therefore, we obtain

$$\|\varphi_{\alpha\beta}\|_{L^\infty(S_{ij})} \le \frac{H_i^2 H_j^2}{(h_{min}^{x_1})^2 (h_{min}^{x_2})^2} \le C,$$

where in the last inequality we used that the ratios $H_i/h_{min}^{x_1}$, $H_j/h_{min}^{x_2}$ are uniformly bounded because the ratios h_{i+1}/h_i, h_{j+1}/h_j are as well (see Remark 2.5).

Summing up we conclude that

$$||I_2 v||_{L^\infty(S_{ij})} \le \sum_{\alpha,\beta=1,2,3} ||\varphi_{\alpha\beta}||_{L^\infty(S_{ij})} ||v||_{L^\infty(S_{ij})} \le C||v||_{L^\infty(S_{ij})}$$

as we wanted to prove. □

Lemma 4.2. *There exists a constant C such that, for any $v \in V_N$,*

$$\left\| \tfrac{\partial I_2 v}{\partial x_1} \right\|_{L^\infty(S_{ij})} \le C \left\| \tfrac{\partial v}{\partial x_1} \right\|_{L^\infty(S_{ij})},$$

$$\left\| \tfrac{\partial I_2 v}{\partial x_2} \right\|_{L^\infty(S_{ij})} \le C \left\| \tfrac{\partial v}{\partial x_2} \right\|_{L^\infty(S_{ij})}.$$

Proof. Let us prove the first inequality. Clearly, analogous arguments apply to obtain the second one.

Using expressions (61) for $I_2 v$ and (60) for $\varphi_{\alpha\beta}$, we obtain that

$$\begin{aligned}\frac{\partial I_2 v}{\partial x_1}(x_1, x_2) &= \sum_{\beta=1,2,3} \sum_{\alpha=1,2,3} \frac{\partial \varphi_{\alpha\beta}}{\partial x_1}(x_1, x_2) v(x_1^\alpha, x_2^\beta) \\ &= \sum_{\beta=1,2,3} \prod_{\ell\neq\beta} \frac{x_2 - x_2^\ell}{x_2^\beta - x_2^\ell} \sum_{\alpha=1,2,3} \frac{\partial}{\partial x_1} \left(\prod_{k\neq\alpha} \frac{x_1 - x_1^k}{x_1^\alpha - x_1^k} \right) v(x_1^\alpha, x_2^\beta).\end{aligned}$$

We observe that the ratios involving the x_2 variable can be bounded by a constant, as in the previous proof. On the other hand, for each β, let $p_\beta(x_1)$ be the quadratic interpolant of $v_\beta := v(\cdot, x_2^\beta)$ over the points x_1^α, $\alpha = 1, 2, 3$, that is

$$p_\beta(x_1) = \sum_{\alpha=1,2,3} \prod_{k\neq\alpha} \frac{x_1 - x_1^k}{x_1^\alpha - x_1^k} v(x_1^\alpha, x_2^\beta)$$

and therefore

$$\frac{\partial I_2 v}{\partial x_1}(x_1, x_2) = \sum_{\beta=1,2,3} \prod_{\ell\neq\beta} \frac{x_2 - x_2^\ell}{x_2^\beta - x_2^\ell} p'_\beta(x_1).$$

Since we can also write

$$p_\beta(x_1) = v_\beta(x_1^1) + \tfrac{v_\beta(x_1^2) - v_\beta(x_1^1)}{x_1^2 - x_1^1}(x_1 - x_1^1) + \frac{\frac{v_\beta(x_1^3)-v_\beta(x_1^2)}{x_1^3-x_1^2} - \frac{v_\beta(x_1^2)-v_\beta(x_1^1)}{x_1^2-x_1^1}}{x_1^3 - x_1^1}(x_1 - x_1^1)(x_1 - x_1^2)$$

we have, if $x_1^M = \frac{x_1^1 + x_1^2}{2}$, that

$$p'_\beta(x_1) = \tfrac{v_\beta(x_1^2) - v_\beta(x_1^1)}{x_1^2 - x_1^1} + 2\frac{\frac{v_\beta(x_1^3)-v_\beta(x_1^2)}{x_1^3-x_1^2} - \frac{v_\beta(x_1^2)-v_\beta(x_1^1)}{x_1^2-x_1^1}}{x_1^3 - x_1^1}\left(x_1 - x_1^M\right).$$

Afterwards, by the Mean Value Theorem, there exist $\zeta_0 \in (x_1^1, x_1^2)$ and $\zeta_1 \in (x_1^2, x_1^3)$ such that

$$p'_\beta(x_1) = v'_\beta(\zeta_0) + 2\tfrac{v'_\beta(\zeta_1) - v'_\beta(\zeta_0)}{x_1^3 - x_1^1}(x_1 - x_1^M).$$

Now, remembering that $|x_1^3 - x_1^1| = H_i$ and $|x_1 - x_1^M| \leq H_i$, we get

$$|p'_\beta(x_1)| \leq |v'_\beta(\zeta_0)| + 2\tfrac{|v'_\beta(\zeta_1)| + |v'_\beta(\zeta_0)|}{H_i} H_i \leq 5 \left\| \tfrac{\partial v}{\partial x_1}(\cdot, x_2^\beta) \right\|_{L^\infty(x_1^1, x_1^3)}.$$

Summing up, we obtain

$$\left\| \tfrac{\partial I_2 v}{\partial x_1} \right\|_{L^\infty(S_{ij})} \leq C \sum_{\beta=1,2,3} \left\| \tfrac{\partial v}{\partial x_1}(\cdot, x_2^\beta) \right\|_{L^\infty(x_1^1, x_1^3)} \leq C \left\| \tfrac{\partial v}{\partial x_1} \right\|_{L^\infty(S_{ij})}$$

as we wanted to show. □

Lemma 4.3. *Let w be the solution of* (51) *and $I_2 w$ its piecewise biquadratic interpolation on $\mathcal{S}_N$. There exists a constant C such that*

$$\|w - I_2 w\|_{\varepsilon^2} \leq Ch^2. \tag{62}$$

Proof. The result follows the a priori estimates provided in Subsection 4.1 and the following interpolation error estimates for the operator I_2 (see [16, Theorem 2.7] and [11, Lemma 4.1]). For $v \in H^3(S_{ij})$, we have

$$\|v - I_2 v\|_{0,S_{ij}} \leq C\left[H_i^2 \left\|\frac{\partial^2 v}{\partial x_1^2}\right\|_{0,S_{ij}} + H_j^2 \left\|\frac{\partial^2 v}{\partial x_2^2}\right\|_{0,S_{ij}}\right], \tag{63}$$

$$\left\|\frac{\partial(v - I_2 v)}{\partial x_1}\right\|_{0,S_{ij}} \leq C\left[H_i^2 \left\|\frac{\partial^3 v}{\partial x_1^3}\right\|_{0,S_{ij}} + H_j^2 \left\|\frac{\partial^3 v}{\partial x_1 \partial x_2^2}\right\|_{0,S_{ij}}\right], \tag{64}$$

$$\left\|\frac{\partial(v - I_2 v)}{\partial x_2}\right\|_{0,S_{ij}} \leq C\left[H_i^2 \left\|\frac{\partial^3 v}{\partial x_1^2 \partial x_2}\right\|_{0,S_{ij}} + H_j^2 \left\|\frac{\partial^3 v}{\partial x_2^3}\right\|_{0,S_{ij}}\right], \tag{65}$$

where the constant C is independent of the element S_{ij} and v. With the notation introduced in Figure 2, we write Ω as

$$\Omega = \bigcup_{i=1}^{8} B_i,$$

where

$$B_1 = \bigcup_{j=1}^{N} S_{1j}, \quad B_2 = \bigcup_{i=1}^{N} S_{i1}, \quad B_3 = \bigcup_{j=1}^{N} S_{Nj}, \quad B_4 = \bigcup_{i=1}^{N} S_{iN},$$

and

$$B_5 = \bigcup \{S_{ij} : 2 \leq i,\, j \leq N/2 - 1\},$$
$$B_6 = \bigcup \{S_{ij} : N/2 \leq i \leq N - 1,\, 2 \leq j \leq N/2 - 1\},$$
$$B_7 = \bigcup \{S_{ij} : N/2 \leq i,\, j \leq N - 1\},$$
$$B_8 = \bigcup \{S_{ij} : 2 \leq i \leq N/2 - 1,\, N/2 \leq j \leq N - 1\}.$$

Due to the symmetry of the problem, it is sufficient to estimate (62) over B_1 and B_5. Starting with B_1, we have

$$\|w - I_2 w\|^2_{\varepsilon^2,\, B_1} = \varepsilon^2 \|\nabla(w - I_2 w)\|^2_{0,\, B_1} + \|w - I_2 w\|^2_{0,\, B_1}. \tag{66}$$

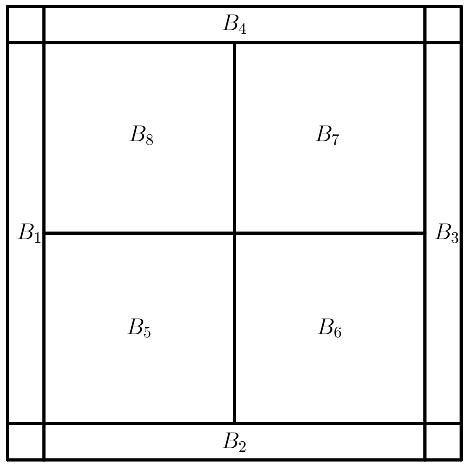

Figure 2: Split of the unitary square domain used in the proof of Lemma 4.3

Here we obtain, for the first term on the right side

$$\varepsilon^2 \|\nabla(w - I_2 w)\|_{0,B_1}^2 = \varepsilon^2 \left\| \frac{\partial(w - I_2 w)}{\partial x_1} \right\|_{0,B_1}^2 + \varepsilon^2 \left\| \frac{\partial(w - I_2 w)}{\partial x_2} \right\|_{0,B_1}^2 . \qquad (67)$$

From (54) and (55), we can observe that

$$\left| \frac{\partial w}{\partial x_i} \right| \leq \frac{C}{\varepsilon}, \qquad i = 1, 2.$$

Taking this into account, using Lemma 4.2 and since $|B_1| = H_1 = Ch^{\frac{1}{1-\eta}}$, we obtain

$$\varepsilon^2 \left\| \frac{\partial(w - I_2 w)}{\partial x_1} \right\|_{0,B_1}^2 \leq Ch^{\frac{1}{1-\eta}}.$$

A similar result is obtained for the derivative with respect to x_2 of the interpolation error and, therefore, we deduce from (67) that

$$\varepsilon^2 \|\nabla(w - I_2 w)\|_{0,B_1}^2 \leq Ch^{\frac{1}{1-\eta}}. \qquad (68)$$

Similarly, using Lemma 4.1 and taking into account that w is uniformly bounded, we have

$$\|w - I_2 w\|_{0,B_1}^2 \leq Ch^{\frac{1}{1-\eta}}. \qquad (69)$$

Finally, with $\eta \geq \frac{3}{4}$ in (68) and (69), it follows

$$\|w - I_2 w\|_{\varepsilon^2,\, B_1}^2 \leq Ch^4.$$

Now we deal with the estimate on B_5. We consider S_{ij} for $2 \leq i, j \leq N/2 - 1$, then

$$\|w - I_2 w\|_{\varepsilon^2,\, S_{ij}}^2 = \varepsilon^2 \|\nabla(w - I_2 w)\|_{0,\, S_{ij}}^2 + \|w - I_2 w\|_{0,\, S_{ij}}^2. \tag{70}$$

We have again for the first term on the right side

$$\varepsilon^2 \|\nabla(w - I_2 w)\|_{0,\, S_{ij}}^2 = \varepsilon^2 \left\| \frac{\partial(w - I_2 w)}{\partial x_1} \right\|_{0,\, S_{ij}}^2 + \varepsilon^2 \left\| \frac{\partial(w - I_2 w)}{\partial x_2} \right\|_{0,\, S_{ij}}^2. \tag{71}$$

We remark that the lengths H_i, H_j of the elements S_{ij} considered here satisfy $H_i \leq Chx_1^\eta$, $H_j \leq Chx_2^\eta$ for $(x_1, x_2) \in S_{ij}$. Using this in equation (64) we have

$$\left\| \frac{\partial(w - I_2 w)}{\partial x_1} \right\|_{0,\, S_{ij}}^2 \leq Ch^4 \left[\left\| x_1^{2\eta} \frac{\partial^3 w}{\partial x_1^3} \right\|_{0,\, S_{ij}}^2 + \left\| x_2^{2\eta} \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_{0,\, S_{ij}}^2 \right] \leq Ch^4 \varepsilon^{-2}$$

where the last inequality is a consequence of (56), (57) and the condition $\eta \geq \frac{3}{4}$. Since the corresponding inequality

$$\left\| \frac{\partial(w - I_2 w)}{\partial x_2} \right\|_{0,\, S_{ij}}^2 \leq Ch^4 \varepsilon^{-2}$$

is proved analogously, we obtain

$$\varepsilon^2 \|\nabla(w - I_2 w)\|_{0,\, S_{ij}}^2 \leq Ch^4. \tag{72}$$

Similarly, for the second term on the right side of (70), using (63) and then (56), with $\eta \geq \frac{3}{4}$, we have

$$\|(w - I_2 w)\|_{0,\, S_{ij}}^2 \leq Ch^4 \left[\left\| x_1^{2\eta} \frac{\partial^2 w}{\partial x_1^2} \right\|_{0,\, S_{ij}}^2 + \left\| x_2^{2\eta} \frac{\partial^2 w}{\partial x_2^2} \right\|_{0,\, S_{ij}}^2 \right] \leq Ch^4. \tag{73}$$

Finally, adding inequalities (72) and (73) on $S_{ij} \in B_5$,

$$\|w - I_2 w\|_{\varepsilon^2, B_5}^2 \leq Ch^4$$

concluding the proof. □

The proof of the next Lemma follows by the same arguments used in [11, Lemma 4.2].

Lemma 4.4. *There exists a constant C such that, for any $v \in V_N$,*

$$\|I_2 v\|_{\varepsilon^2} \leq C\|v\|_{\varepsilon^2}. \tag{74}$$

Proposition 4.1. *Let w be the solution of* (51)*, $w_N \in V_N$ its finite element approximation and $w_N^* = I_2 w_N$. Suppose that $\frac{3}{4} \leq \eta < 1$. Then, there exists a constant C such that,*

$$\|w - w_N^*\|_{\varepsilon^2} \leq Ch^2 \log^{\frac{1}{2}}\left(\tfrac{1}{\varepsilon}\right). \tag{75}$$

Proof. Since $I_2 w_I = I_2 w$, we have

$$\|w - w_N^*\|_{\varepsilon^2} \leq \|w - I_2 w\|_{\varepsilon^2} + \|I_2(w_I - w_N)\|_{\varepsilon^2}$$

and therefore, combining (62), (74) and (58), we conclude the proof. □

Remark 4.2. *With the assumed regularity and compatibility condition of f, the estimates* (54)*,* (55)*,* (56) *and* (57)*, are also valid in the non-singularly perturbed case (moderate values of ε). Therefore, the superconvergence result of Proposition 4.1 holds in that case as well. We will use this fact in the next Section to obtain our main result.*

5. The error estimate

We recall the functions $U_i \in H_0^1(\Omega)$, associated with the eigenvalue μ_i, solutions of the variational problems (14), and their piecewise bilinear approximations $U_{i,N} \in V_N$ introduced by (24) with the space V_N defined in (23). Let $U_{i,N}^*$ be the post-processed of $U_{i,N}$ introduced in the previous Section. According to Proposition 4.1 with $\varepsilon = \sqrt{\mu_i}$ and $h = \frac{1}{N}$, it holds

$$\|U_i - U_{i,N}^*\|_{\mu_i} \leq C\frac{1}{N^2}\left|\log \mu_i\right|^{\frac{1}{2}} (\log M)^s \tag{76}$$

with C depending on f and s, but independent of N, M and μ_i. We have used the estimate $|v_i(0)| \leq (\log M)^s$ from Remark 2.4.

Now we choose $N = M^{\frac{3}{4}}$. Then inserting (35) (incorporating $\|f\|_0$ to the constant C) and (31) into (30), and using (76) for $i = 1, 2, \ldots, M$, we obtain

$$\begin{aligned}
\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} &\lesssim (\log M)^{\frac{3+\sigma\alpha}{2}} M^{-1} + \left(\sum_{i=1}^{M} |\log \mu_i| (\log M)^{2s} N^{-4} \right)^{\frac{1}{2}} \\
&\leq C M^{-1} \left[(\log M)^{\frac{3+\sigma\alpha}{2}} + (\log M)^{s} \left(\max_{1\leq i\leq M} |\log \mu_i| \right)^{\frac{1}{2}} \right] \\
&\leq C M^{-1} \left[(\log M)^{\frac{3+\sigma\alpha}{2}} + (\log M)^{\frac{1}{2}+s} \right] \\
&\leq C M^{-1} (\log M)^{\max\left(\frac{3+\sigma\alpha}{2}, \frac{1}{2}+s\right)}
\end{aligned} \tag{77}$$

where we used, taking into account the upper and lower bounds (15) and (16) for the eigenvalues μ_i, that

$$\max_{1\leq i\leq M} |\log \mu_i| \leq C \log M.$$

Then we have proven our main Theorem which we can now state.

Theorem 5.1. *Let f be a $\mathcal{C}^2(\overline{\Omega})$ function satisfying the compatibility condition* (53)*, with $\Omega = (0,1)^2$. Let $s \in (0,1)$. Given $M \in \mathbb{N}$, let $N = M^{\frac{3}{4}}$. We consider the approximation $u_{M,N}$ given by* (26)*, with the grid $\mathcal{G}^M$ introduced in Subsection 2.4 with the parameter σ satisfying* (17) *and the graded mesh $\mathcal{T}_N$ defined with η satisfying* (19)*. Then there exists a constant C, depending only on s and f, such that*

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C (\log M)^t M^{-1}, \tag{78}$$

with $t = \max\left(\frac{3+\sigma\alpha}{2}, \frac{1}{2}+s\right)$.

We can rewrite the result in terms of the total number N_{dof} of degrees of freedom. Notice that our discretization requires to solve M reaction diffusion problems each one of them having $O(N^2)$ degrees of freedom. It follows that $N_{dof} \sim M^{\frac{5}{2}}$. Therefore we can rewrite (78) as

$$\|u - u_{M,N}\|_{\mathbb{H}^s(\Omega)} \leq C (\log N_{dof})^t N_{dof}^{-\frac{2}{5}}.$$

This order of convergence is suboptimal, since for a regular two dimensional problem an error of order $N_{dof}^{-\frac{1}{2}}$ is expected, but it is a little better than the result obtained in [6, Theorem 3] where, additionally, a stronger boundary compatibility is assumed on f. On the other hand, we would like to emphasize that the M linear systems coming from the approximation of the reaction–diffusion problems (24) have a simple structure, they are of the form

$$\mu_i A_1 + A_0 = d_s v_i(0) b \qquad i = 1, \ldots, M$$

with the matrices A_0 and A_1 and the vector b depending only on the graded mesh $\mathcal{T}_N$. Then the M linear systems can be simultaneously obtained one time the eigenpairs (μ_i, v_i) are known, and then parallelization algorithms could be applied to improve the performance.

6. Numerical examples

In order to confirm the results of Theorem 5.1 we approximate the solution of problem

$$\begin{aligned} (-\Delta)^s u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{79}$$

in two examples. The computations were implemented using GNU Octave [17]. The eigenproblems of Subsection 2.3 were solved using the command `eig`, and the linear problems (24) were solved with Octave's *backslash* "\" operator. Taking into account the symmetry of the problems, the errors are computed on the subdomain $[0, \frac{1}{2}]$ in the 1d case or on $[0, \frac{1}{2}]^2$ on the 2d cases.

We measure the error in the energy norm $\|\cdot\|_s$ which is estimated by

$$\int_\Omega |f(u - u_{M,N})|$$

since

$$\|u - u_{M,N}\|_s^2 \lesssim \|\mathcal{U} - \mathcal{U}_{M,N}\|^2_{L^2(y^\alpha, \mathcal{C})} = d_s \int_\Omega f(u - u_{M,N}).$$

Example 1. We consider problem (79) with $\Omega = [0, 1]^2$ and

$$f(x, y) = (x + y)(x + y - 2)((x - y)^2 - 1).$$

For the exponents $s = 0.25$ and $s = 0.75$ we approximate the problem as stated in Subsection 2.4 with the following parameters

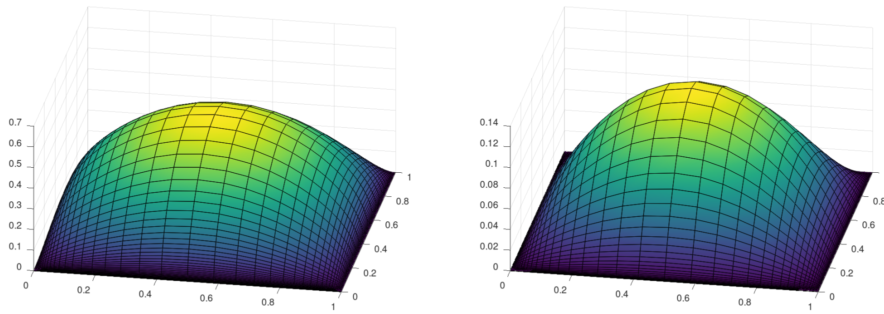

Figure 3: Numerical solutions for Example 1 with $s = 0.25$ (left) and $s = 0.75$ (right)

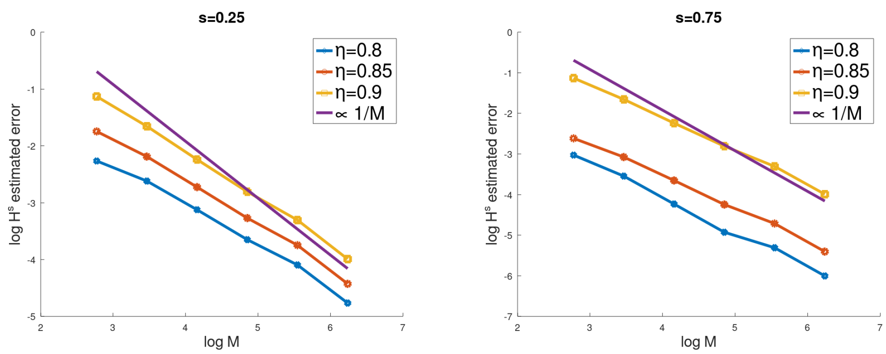

Figure 4: Numerical errors for the Example 1

- $\sigma = \frac{1-0.9s}{1+0.1s}$, which implies $\frac{1}{1-\sigma} = \frac{1}{s} + 0.1$,
- $\mathcal{Y} = 2 \log M$,
- η varies in $\{0.8, 0.85, 0.9\}$,
- M is taken as $M = 2^i$, with $i = 4, 5, \ldots, 9$.

Since the exact solutions are not known, the numerical errors were estimated by comparing with a solution obtained for the largest value of $M = 1024$.

We show pictures of the solutions in Figure 3 obtained for $M = 256$. In Figures 4, we plot the estimated H^s error versus M in logarithmic scale, observing an order of convergence close to 1, which confirms the results in Theorem 5.1.

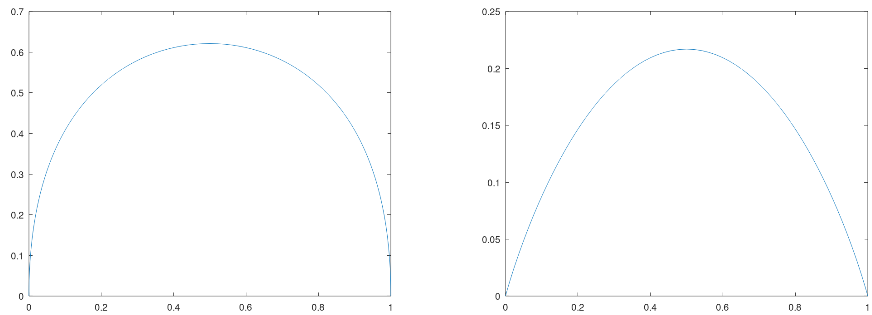

Figure 5: Numerical solutions for Example 2, 1d–case, with $s = 0.25$ (left) and $s = 0.75$ (right)

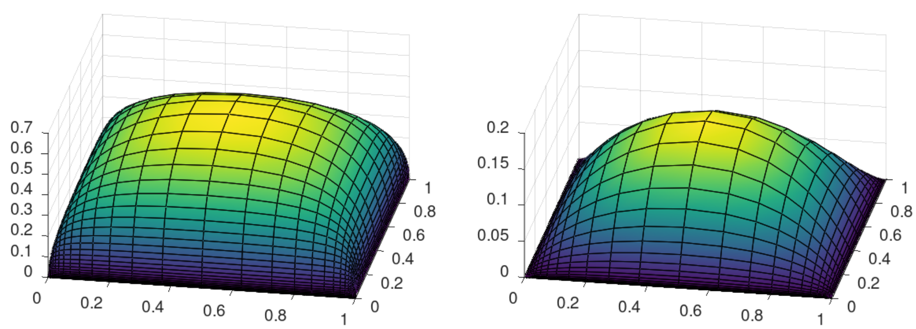

Figure 6: Numerical solutions for Example 2, 2d–case, with $s = 0.25$ (left) and $s = 0.75$ (right)

Example 2. In this Example we take a right hand side f which do not satisfies the compatibility condition (53), in two cases:

- the 1d–case: $\Omega = (0, 1)$, and $f(x) = 1$,
- the 2d–case: $\Omega = (0, 1)^2$, and $f(x, y) = 1$.

Figures 5 and 6 show pictures of the solutions obtained for $M = 256$.

In Figures 7 and 8, we plot the estimated H^s error versus M in logarithmic scale. We observe that, although this example is not covered by the theory, the results are according with Theorem 5.1. We also show in Figure 9 the errors obtained when using uniform meshes for the discretization of the singular reaction diffusion problems in the 2D-case, with the same number

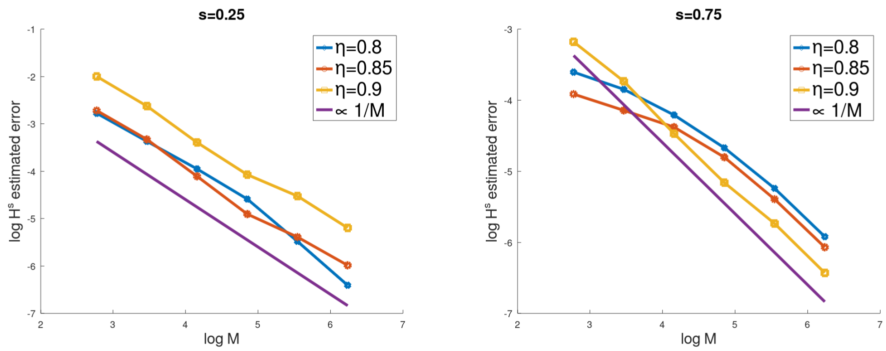

Figure 7: Numerical errors Example 2, 1d–case

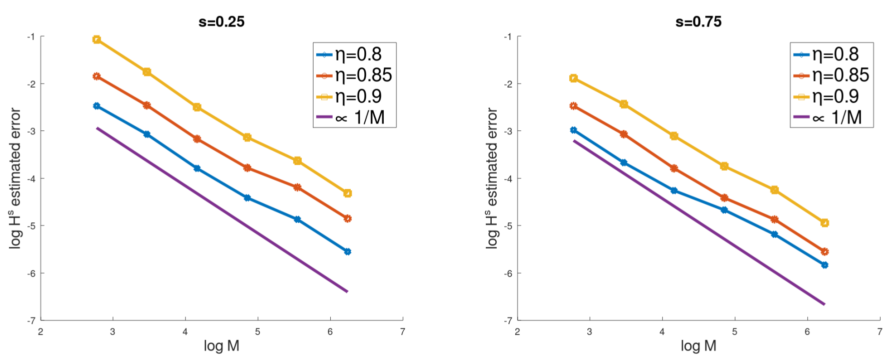

Figure 8: Numerical errors for Example 2, 2d–case

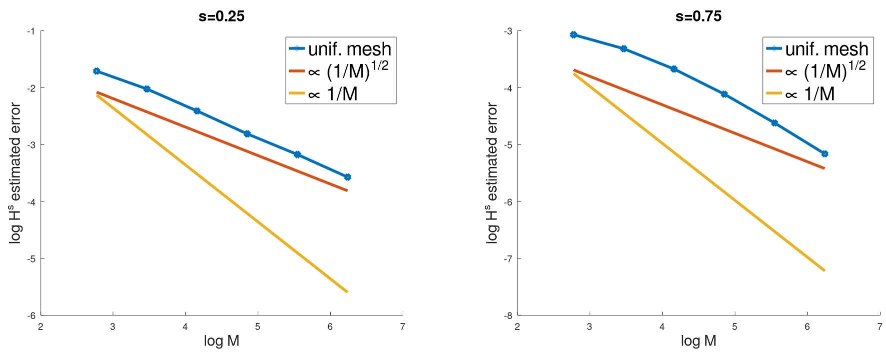

Figure 9: Numerical errors for Example 2, 2d–case, using uniform meshes on Ω

of elements as in the corresponding graded meshes cases. In this case, we observe that the order of convergence is reduced, becoming closer to $\frac{1}{2}$.

Acknowledgements

This work was partially supported by Universidad Nacional de Rosario under grant 80020220700222U. Additionally the second (A.L. Lombardi) and third (C. Penessi) authors were supported by Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación (Argentina) under grant PICT 2018-3017.

References

[1] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the laplacian, Advances in Mathematics 224 (5) (2010) 2052–2093.

[2] L. Caffarelli, L. Silvestre, An extension problem related to the fractional laplacian, Communications in partial differential equations 32 (8) (2007) 1245–1260.

[3] D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, Application of a fractional advection-dispersion equation, Water resources research 36 (6) (2000) 1403–1412.

[4] P. R. Stinga, J. L. Torrea, Extension problem and harnack's inequality for some fractional operators, Communications in Partial Differential Equations 35 (11) (2010) 2092–2122.

[5] R. H. Nochetto, E. Otárola, A. J. Salgado, A pde approach to fractional diffusion in general domains: a priori error analysis, Foundations of Computational Mathematics 15 (2015) 733–791.

[6] L. Banjai, J. M. Melenk, R. H. Nochetto, E. Otárola, A. J. Salgado, C. Schwab, Tensor fem for spectral fractional diffusion, Foundations of Computational Mathematics 19 (2019) 901–962.

[7] R. G. Durán, A. L. Lombardi, Error estimates on anisotropic $\mathcal{Q}_1$ elements for functions in weighted Sobolev spaces, Math. Comput. 74 (252) (2005) 1679–1706. doi:10.1090/S0025-5718-05-01732-1.

[8] R. G. Durán, A. L. Lombardi, M. I. Prieto, Supercloseness on graded meshes for q1 finite element approximation of a reaction–diffusion equation, Journal of computational and applied mathematics 242 (2013) 232–247.

[9] T. Linß, Uniform superconvergence of a galerkin finite element method on shishkin-type meshes, Numerical Methods for Partial Differential Equations: An International Journal 16 (5) (2000) 426–440.

[10] M. Stynes, L. Tobiska, The sdfem for a convection-diffusion problem with a boundary layer: optimal error analysis and enhancement of accuracy, SIAM journal on numerical analysis 41 (5) (2003) 1620–1642.

[11] R. G. Durán, A. L. Lombardi, M. I. Prieto, Superconvergence for finite element approximation of a convection-diffusion equation using graded meshes, IMA J. Numer. Anal. 32 (2) (2012) 511–533. doi:10.1093/imanum/drr005.
URL semanticscholar.org/paper/e83d1dfbb8bb6a879ff15378e44c2cf17d38d649

[12] Z. Zhang, Finite element superconvergence approximation for one-dimensional singularly perturbed problems, Numer. Methods Partial Differ. Equations 18 (3) (2002) 374–395. doi:10.1002/num.10001.

[13] Z. Zhang, Finite element superconvergence approximation for one-dimensional singularly perturbed problems, Numer. Methods Partial Differ. Equations 18 (3) (2002) 374–395. doi:10.1002/num.10001.

[14] J. Li, M. F. Wheeler, Uniform convergence and superconvergence of mixed finite element methods on anisotropically refined grids,

SIAM J. Numer. Anal. 38 (3) (2000) 770–798. doi:10.1137/S0036142999351212.

[15] R. G. Duran, M. A. Muschietti, R. Rodriguez, Asymptotically exact error estimators for rectangular finite elements, SIAM journal on numerical analysis 29 (1) (1992) 78–88.

[16] T. Apel, Anisotropic finite elements: Local estimates and applications, Leipzig: Teubner; Chemnitz: Technische Univ., 1999.

[17] J. W. Eaton, D. Bateman, S. Hauberg, R. Wehbring, GNU Octave version 8.4.0 manual: a high-level interactive language for numerical computations (2023).
URL https://www.gnu.org/software/octave/doc/v8.4.0/