Superconvergence for finite element approximation of a convection-diffusion equation using graded meshes *

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Abstract. In this paper we analyze the approximation of a model convection-diffusion equation by standard bilinear finite elements using the graded meshes introduced in [5].
Our main goal is to prove superconvergence results of the type known for standard elliptic problems, namely, that the difference between the finite element solution and the Lagrange interpolation of the exact solution, in the ε-weighted H^1-norm, is of higher order than the error itself. The constant in our estimate depends only weakly on the singular perturbation parameter.
As a consequence of the superconvergence result, we obtain optimal order error estimates in the L^2-norm. Also, we show how to obtain a higher order approximation by a local postprocessing of the computed solution.

Key words. Convection-diffusion, superconvergence, graded meshes.

AMS subject classifications. 65N30.

1 Introduction

This paper deals with the approximation of a convection-diffusion problem by standard $\mathcal{Q}_1$ rectangular finite elements. For convection dominated problems it is well known that standard finite elements produce poor approximations unless very fine or appropriate meshes are used.
In some cases, for example in the model problem considered here, one can use known behavior of the exact solution to design a priori adapted meshes to approximate well the boundary layer. A lot of work has been done in this direction. Probably the most well known approximations of this kind are those based on the so called Shishkin meshes. In particular, optimal order of convergence have been proved when Shishkin meshes are used in combination with standard finite elements or some stream line artificial diffusion methods (see for example [10, 1, 12]).
More recently, in [4, 5], the use of graded meshes for reaction-diffusion and convection-diffusion problems was analyzed and almost optimal error estimates were obtained.
On the other hand, superconvergence for elliptic problems with smooth solutions has been developed in a lot of papers since the work of Zlamal [19] (see for example the book [16]). For convection-diffusion and reaction-diffusion problems, superconvergence results for approximations based on the use of Shishkin meshes have been proved in [8, 14, 17, 18].
In this paper we analyze whether similar results than those obtained for Shishkin type meshes are valid for graded meshes. This kind of meshes have been introduced as an alternative to the Shishkin ones. Numerical experiments with both kind of meshes seems to show a similar behavior of the error. As we are going to show, superconvergence results are valid also for graded meshes. Our results are slightly weaker than those obtained previously for Shishkin meshes because of logarithmic factors of ε involved in our estimates. However, the graded meshes have some desirable

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properties which the Shishkin meshes do not satisfy. Indeed, when one is approximating a singularly perturbed problem with an a priori adapted mesh, it is natural to expect that a mesh designed for some value of the perturbation parameter ε work well also for larger values of it. It was shown in [4] that this is the case for the meshes introduced in that paper but not for the Shishkin meshes. We will present some numerical results showing that the same is true for superconvergence. This fact can be important in problems where the diffusion parameter is not constant or, also, to treat systems of equations in which different equations have singular perturbations of different orders. Let us mention that, for this kind of systems, Shishkin type meshes have been used in [9] (see also [15] where a similar method is used for initial value problems). In that paper, the authors modify the classic two part Shishkin meshes dividing the domain in several parts and dividing uniformly each one of these parts. One can see that in this way one obtains something intermediate between the usual Shishkin meshes and the graded ones.

We will prove superconvergence error estimates for the standard $\mathcal{Q}_1$ finite element approximation of a model convection-diffusion problem when graded meshes are used. Precisely, if u_h is the finite element solution (where h is a parameter related with the definition of the meshes) and u_I is the Lagrange interpolation of the exact solution u, we prove that $\|u_I - u_h\|_\varepsilon$ is of higher order than $\|u - u_h\|_\varepsilon$, where $\|.\|_\varepsilon$ is the ε-weighted H^1-norm associated with the symmetric part of the differential equation. This result, combined with interpolation error estimates obtained in [5], gives optimal order convergence in the L^2 norm. Both superconvergence in the $\|.\|_\varepsilon$ norm as well as optimal order convergence in the L^2 norm are almost optimal, in the sense that the constants depend only on the logarithm of the singular perturbation parameter. Our arguments combine ideas of [5, 18, 19].

As an application of our superconvergence error estimate, we show how to obtain a higher order approximation by a simple local postprocessing of the computed solution. Let us mention that postprocessing procedures for convection-diffusion equations using Shishkin type meshes have been given in [11, 14].

We consider the model problem,

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where ε is a small positive parameter, and $\Omega = [0,1]^2$. We assume that $b = (b_1, b_2)$, c and f are smooth on Ω and that

$$b_i < -\gamma, \quad \text{with } \gamma > 0 \quad \text{for } i = 1,2.$$

Then, the solution will have a boundary layer of width $\mathcal{O}(\varepsilon \log(1/\varepsilon))$ at the outflow boundary $\{(x,y) \in \partial\Omega : x = 0 \text{ or } y = 0\}$ (see [12]). Moreover, we will make the usual assumption in order to have coerciveness of the bilinear form associated with Problem 1.1, namely, there exists a constant μ such that

$$c - \frac{1}{2}\operatorname{div}(b) \geq \mu > 0. \tag{1.2}$$

In our proofs we will need some weighted *a priori* estimates for the solution u. To prove these estimates we will use that u is sufficiently smooth on Ω and its derivatives satisfy some bounds. In order to have these results it is enough to assume that $f \in \mathcal{C}^4(\Omega)$ and satisfies the following compatibility conditions,

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0,$$

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(1,1) = 0 \text{ for } 0 \leq i+j \leq 3,$$

Indeed, under these conditions it can be proved that Problem 1.1 has a classical solution $u \in \mathcal{C}^3(\Omega)$, and for all $(x,y) \in \Omega$ we have

$$\left| \frac{\partial^{i+j} u}{\partial x^i \partial y^j}(x,y) \right| \le C \left(1 + \varepsilon^{-i} e^{-\gamma x/\varepsilon} + \varepsilon^{-j} e^{-\gamma y/\varepsilon} + \varepsilon^{-(i+j)} e^{-\gamma x/\varepsilon} e^{-\gamma y/\varepsilon} \right) \tag{1.3}$$

for $0 \le i + j \le 3$. We refer to [11, Section 2] and [10] for details.
Here, and in the rest of the paper, the letter C denotes a generic constant independent of ε and of the discretization parameter h.
The rest of the paper is organized as follows. In Section 2 we present the graded meshes and some preliminary results. Section 3 contains our main results concerning the superconvergence error estimates. In Section 4 we show how a higher order approximation can be obtained from the computed solution by a simple local postprocessing, and finally, in Section 5 we present some numerical results.

2 Finite element approximation on graded meshes

In this section we recall the graded meshes used in [5] and prove some preliminary results for our error analysis.
For a domain D we use the standard notation for Sobolev spaces, norms and seminorms, namely,

$$\|u\|_{m,D} := \left\{ \sum_{\alpha \le m} \|\mathcal{D}^\alpha u\|^2_{L^2(\Omega)} \right\}^{1/2}, \qquad |u|_{m,D} := \left\{ \sum_{\alpha = m} \|\mathcal{D}^\alpha u\|^2_{L^2(\Omega)} \right\}^{1/2}.$$

In particular $\|u\|_{0,D}$ denotes the L^2-norm of u. When $D = \Omega$, and no confusion can arise, we will write $\|u\|_0$ instead of $\|u\|_{0,\Omega}$.
For a rectangle R, $\mathcal{P}_k(R)$ and $\mathcal{Q}_k(R)$ denote the spaces of polynomials of total degree less than or equal to k and of degree less than or equal to k in each variable respectively, over R.
The standard weak formulation of Problem 1.1 is given by

$$\mathcal{B}(u,v) = \int_\Omega f\, v\, dx \qquad \forall v \in H^1_0(\Omega), \tag{2.1}$$

where the bilinear form $\mathcal{B}$ is defined as

$$\mathcal{B}(u,v) = \int_\Omega (\varepsilon \nabla u \cdot \nabla v + b \cdot \nabla u\, v + cuv)\, dx. \tag{2.2}$$

We will work with the ε-weighted H^1-norm defined by

$$\|v\|^2_\varepsilon = \varepsilon \|\nabla v\|^2_0 + \|v\|^2_0 .$$

It is well known that, under the hypothesis (1.2), the bilinear form $\mathcal{B}$ is coercive in the ε-norm, moreover, there exists $\beta > 0$, independent of ε, such that

$$\beta \|v\|_\varepsilon \le \mathcal{B}(v,v) \quad \forall v \in H^1_0(\Omega). \tag{2.3}$$

However, the continuity of $\mathcal{B}$ is not uniform in ε, and therefore, the standard theory based on Cea's lemma can not be applied to obtain error estimates valid uniformly in ε.
In [5] an analysis for the approximation of Problem (1.1) by standard bilinear finite elements, using appropriate graded meshes, was developed. Almost optimal order of convergence independent of ε was proved in that paper. Here we will prove superconvergence for the same approximation considered in [5].

First let us recall the graded meshes used in [5]. Given the discretization parameter h, that we suppose $0 < h < 1$, consider the partition $\{\xi_i\}_{i=0}^M$ of the interval $[0,1]$ given by

$$\left\{\begin{array}{rcll} \xi_0 & = & 0 & \\ \xi_1 & = & h\varepsilon & \\ \xi_{i+1} & = & \xi_i + h\xi_i & \text{for } 1 \leq i \leq M-2 \\ \xi_M & = & 1 & \end{array}\right. \tag{2.4}$$

where M is such that $\xi_{M-1} < 1$ and $\xi_{M-1} + h\xi_{M-1} \geq 1$. If $1 - \xi_{M-1} < \xi_{M-1} - \xi_{M-2}$ we modify the definition of ξ_{M-1} taking $\xi_{M-1} = (1 + \xi_{M-2})/2$. In this way we avoid the case in which the last interval is smaller than the previous one.

Remark 2.1. *In practice it is natural to take $h_i := \xi_i - \xi_{i-1}$ to be monotonically increasing. To have this property one can modify the partition by taking $h_i = h_1$ for i such that $\xi_{i-1} < \varepsilon$ and starting with the graded mesh after that. It is not difficult to check that all our arguments can be extended to this case.*

We define $R_{ij} = [\xi_{i-1}, \xi_i] \times [\xi_{j-1}, \xi_j]$, and the graded meshes $\mathcal{T}_h = \{R_{ij}\}_{i,j=1}^M$ on Ω.
Associated with $\mathcal{T}_h$ we introduce the standard piecewise bilinear finite element space

$$V_h = \left\{v \in \mathcal{C}(\Omega) : v\,|_{R_{ij}} \in \mathcal{Q}_1(R_{ij}), 1 \leq i, j \leq M\right\},$$

and the finite element approximation $u_h \in V_h$ given by

$$\mathcal{B}(u_h, v) = \int_\Omega f v \, dx \quad \forall v \in V_h.$$

For the proof of our estimates, we will need to decompose Ω as $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where Ω_1, Ω_2 and Ω_3 are the closed sets with disjoint interiors defined by

$$\begin{aligned} \Omega_1 &= \bigcup \{R_{ij} : \xi_{i-1} < c_1\varepsilon \log(1/\varepsilon)\} \\ \Omega_2 &= \bigcup \{R_{ij} : \xi_{i-1} \geq c_1\varepsilon \log(1/\varepsilon), \xi_{j-1} < c_1\varepsilon \log(1/\varepsilon)\} \\ \Omega_3 &= \bigcup \{R_{ij} : \xi_{i-1} \geq c_1\varepsilon \log(1/\varepsilon), \xi_{j-1} \geq c_1\varepsilon \log(1/\varepsilon)\} \end{aligned}$$

where the constant c_1 is such that

$$\left|\frac{\partial^{i+j} u}{\partial x^i \partial y^j}\right| \leq C \text{ for } 0 \leq i + j \leq 3, \text{ if } x, y > c_1\varepsilon \log(1/\varepsilon). \tag{2.5}$$

Note that, in view of (1.3), it is enough to take $c_1 > 3/\gamma$.
In the following lemma we give several weighted a priori estimates for the solution of Problem 1.1. Some of these estimates were proved in [5] and all of them are consequences of (1.3).

Lemma 2.2. *There exists a constant C such that*

$$\varepsilon^\alpha \left\| y^\beta \frac{\partial^2 u}{\partial y^2} \right\|_{0,\Omega}, \; \varepsilon^\alpha \left\| x^\beta \frac{\partial^2 u}{\partial x^2} \right\|_{0,\Omega} \le C \quad \textit{for} \quad \alpha+\beta \ge 3/2,\ \alpha \ge 0, \beta > -1/2 \tag{2.6}$$

$$\varepsilon^\alpha \left\| y^\beta \frac{\partial^2 u}{\partial x \partial y} \right\|_{0,\Omega}, \; \varepsilon^\alpha \left\| x^\beta \frac{\partial^2 u}{\partial x \partial y} \right\|_{0,\Omega} \le C \quad \textit{for} \quad \alpha+\beta \ge 1,\ \alpha \ge 1/2, \beta > -1/2 \tag{2.7}$$

$$\varepsilon^\alpha \left\| y^\beta \frac{\partial^3 u}{\partial y^3} \right\|_{0,\Omega}, \; \varepsilon^\alpha \left\| x^\beta \frac{\partial^3 u}{\partial x^3} \right\|_{0,\Omega} \le C \quad \textit{for} \quad \alpha+\beta \ge 5/2,\ \alpha \ge 0, \beta > -1/2 \tag{2.8}$$

$$\varepsilon^\alpha \left\| y^\beta \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,\Omega}, \; \varepsilon^\alpha \left\| x^\beta \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,\Omega} \le C \quad \textit{for} \quad \alpha+\beta \ge 2,\ \alpha \ge 1/2, \beta > -1/2 \tag{2.9}$$

$$\varepsilon^\alpha \left\| y^\beta \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,\Omega}, \; \varepsilon^\alpha \left\| x^\beta \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,\Omega} \le C \quad \textit{for} \quad \alpha+\beta \ge 2,\ \alpha \ge 3/2, \beta > -1/2 \tag{2.10}$$

$$\varepsilon^\alpha \left\| xy \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,\Omega}, \; \varepsilon^\alpha \left\| xy \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,\Omega} \le C \quad \textit{for} \quad \alpha \ge 1/2 \tag{2.11}$$

Proof. Let us prove for example one of the inequalities given in (2.8). The other inequalities can be obtained in a similar way.
From (1.3) we have,

$$\int_\Omega x^{2\beta} \left| \frac{\partial^3 u}{\partial x^3} \right|^2 dx\,dy \le C \int_0^1 \int_0^1 x^{2\beta} (1 + \varepsilon^{-6} e^{-2\gamma x/\varepsilon}) dx\,dy$$

and so, integrating first in the variable y and making the change of variable $z = x/\varepsilon$, we obtain

$$\int_\Omega x^{2\beta} \left| \frac{\partial^3 u}{\partial x^3} \right|^2 dx\,dy \le C \left(1 + \varepsilon^{2\beta-5} \int_0^\infty z^{2\beta} e^{-2\gamma z}\, dz \right)$$

and therefore, (2.8) follows easily from the conditions on α and β. □

3 Error estimates

In this section we prove that the finite element approximation defined in the previous section is superconvergent in the ε-weighted H^1-norm, i.e., the difference between the computed solution and the Lagrange interpolation of the exact solution is of higher order than the error itself. In particular, it follows from this result and previously known interpolation error estimates, that the method is almost optimal convergent in the L^2-norm.
Let $x_i = \xi_i$ and $y_j = \xi_j$. With each element $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ we associate the lengths of its edges $h_i = \xi_i - \xi_{i-1}, h_j = \xi_j - \xi_{j-1}$, and we denote with $(\overline{x}_i, \overline{y}_j)$ its barycenter, and with $\ell_k^{i,j}$, for $k = 1, 2, 3, 4$, its edges, as indicated in Figure 1. For a continuous function u, $u_I \in V_h$ denotes the standard piecewise $\mathcal{Q}_1$ Lagrange interpolation of u. We have dropped the dependence on h in the notation u_I to simplify notation.
We will use the following well known results. For $w \in H^2(R_{ij})$, the interpolation satisfies the error estimate (see for example [13]),

$$\left\| \frac{\partial (w - w_I)}{\partial x} \right\|_{0,R_{ij}} \le C \left\{ h_i \left\| \frac{\partial^2 w}{\partial x^2} \right\|_{0,R_{ij}} + h_j \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|_{0,R_{ij}} \right\}. \tag{3.1}$$

Given $u \in H^3(R_{ij})$, there exists $p \in \mathcal{P}_2(R_{ij})$ such that

$$\left\| \frac{\partial^2 (u - p)}{\partial x^2} \right\|_{0,R_{ij}} \le C \left\{ h_i \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}} + h_j \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}} \right\} \tag{3.2}$$

and

$$\left\| \frac{\partial^2 (u-p)}{\partial x \partial y} \right\|_{0,R_{ij}} \le C \left\{ h_i \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}} + h_j \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}} \right\}, \tag{3.3}$$

indeed, we can take p as an averaged Taylor polynomial of u (see for example [2, 3]).

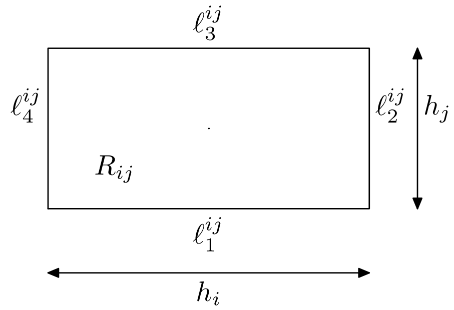


Figure 1: General element

In the following lemma we bound the term corresponding to the diffusion part of the equation. The proof uses an argument introduced by Zlamal in [19].

Lemma 3.1. *Let u be the solution of* (1.1). *There exists a constant C such that, for any $v \in V_h$,*

$$\left| \varepsilon \int_\Omega \nabla(u - u_I) \cdot \nabla v \, dx \, dy \right| \le Ch^2 \|v\|_\varepsilon$$

Proof. Let us prove for example,

$$\left| \varepsilon \int_\Omega \frac{\partial(u-u_I)}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy \right| \le Ch^2 \|v\|_\varepsilon . \tag{3.4}$$

Clearly, analogous arguments apply to estimate the term involving derivatives with respect to y. The key observation made in [19] is that, for $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$,

$$\int_{R_{ij}} \frac{\partial(p-p_I)}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy = 0.$$

Indeed, this follows easily integrating by the midpoint rule and using that $\dfrac{\partial(p-p_I)}{\partial x} \in \mathcal{P}_1(R_{ij})$, vanishes over the segment joining the midpoints of $\ell_1^{i,j}$ and $\ell_3^{i,j}$, and $\dfrac{\partial v}{\partial x} = ay + b$ with $a, b \in \mathbb{R}$. Therefore, for all $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$, we have

$$\begin{aligned} \left| \int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy \right| &= \left| \int_{R_{ij}} \frac{\partial[(u-p)-(u-p)_I]}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy \right| \\ &\le \left\| \frac{\partial[(u-p)-(u-p)_I]}{\partial x} \right\|_{0,R_{ij}} \left\| \frac{\partial v}{\partial x} \right\|_{0,R_{ij}}, \end{aligned}$$

and using (3.1) for $w = u - p$, we obtain

$$\left| \int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy \right| \le C \left\{ h_i \left\| \frac{\partial^2(u-p)}{\partial x^2} \right\|_{0,R_{ij}} + h_j \left\| \frac{\partial^2(u-p)}{\partial x \partial y} \right\|_{0,R_{ij}} \right\} \left\| \frac{\partial v}{\partial x} \right\|_{0,R_{ij}} .$$

Choosing now $p \in \mathcal{P}_2(R_{ij})$ satisfying (3.2) and (3.3) we obtain,

$$\left|\int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x}\frac{\partial v}{\partial x}\,dx\,dy\right|$$
$$\leq C\left\{h_i^2\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}} + h_i h_j\left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,R_{ij}} + h_j^2\left\|\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,R_{ij}}\right\}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{ij}}. \tag{3.5}$$

Let us now estimate the right hand side of (3.5) over each element according to its position. Since $h_1 = \varepsilon h$, we have

$$\left|\varepsilon\int_{R_{11}} \frac{\partial(u-u_I)}{\partial x}\frac{\partial v}{\partial x}\,dx\,dy\right| \leq$$
$$\leq Ch^2\left\{\varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{11}} + \varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,R_{11}} + \varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,R_{11}}\right\}\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{11}}.$$

Now, for $j \geq 2$ and any $(x,y) \in R_{1j}$ we have $h_j \leq hy$, which together with $h_1 = \varepsilon h$ gives

$$\left|\varepsilon\int_{R_{1j}} \frac{\partial(u-u_I)}{\partial x}\frac{\partial v}{\partial x}\,dx\,dy\right| \leq$$
$$\leq Ch^2\left\{\varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{1j}} + \varepsilon^{3/2}\left\|y\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,R_{1j}} + \varepsilon^{1/2}\left\|y^2\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,R_{1j}}\right\}\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{1j}},$$

analogously, for $i \geq 2$, we obtain

$$\left|\varepsilon\int_{R_{i1}} \frac{\partial(u-u_I)}{\partial x}\frac{\partial v}{\partial x}\,dx\,dy\right| \leq$$
$$\leq Ch^2\left\{\varepsilon^{1/2}\left\|x^2\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{i1}} + \varepsilon^{3/2}\left\|x\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,R_{i1}} + \varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,R_{i1}}\right\}\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{i1}}.$$

Finally, for $i, j \geq 2$, using that for any $(x,y) \in R_{ij}$, $h_i \leq hx$ and $h_j \leq hy$, we have

$$\left|\varepsilon\int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x}\frac{\partial v}{\partial x}\,dx\,dy\right| \leq$$
$$\leq Ch^2\left\{\varepsilon^{1/2}\left\|x^2\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}} + \varepsilon^{1/2}\left\|xy\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,R_{ij}} + \varepsilon^{1/2}\left\|y^2\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,R_{ij}}\right\}\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{ij}}, \forall i, j \geq 2.$$

Therefore, summing over all indices i, j, and using the weighted inequalities (2.8), (2.11), and (2.9) given in Lemma 2.2, we obtain (3.4). □

Our next goal is to give an estimate for the term corresponding to the convection. We want to apply an argument similar to that used for the diffusion part. With this goal we define, for $u \in H^3(R_{ij})$ and $v \in V_h$,

$$K_{ij}(u,v) = \int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x}v\,dx\,dy - \frac{h_i^2}{12}\left(\int_{\ell_2^{i,j}} \frac{\partial^2 u}{\partial x^2}v\,dy - \int_{\ell_4^{i,j}} \frac{\partial^2 u}{\partial x^2}v\,dy\right). \tag{3.6}$$

In [7, 18] the authors give an explicit expression of $K_{ij}(u,v)$ (see [18], identity (4.28)) which, in particular, implies the result of our next lemma. We will give a more direct proof without making use of that expression.

Lemma 3.2. *For $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$ we have,*

$$K_{ij}(p, v) = 0$$

Proof. Take $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$, and define $e = p - p_I$. Then, $\frac{\partial e}{\partial x} v$ is a polynomial of degree two in x and of degree one in y which vanishes at the midpoints of $\ell_1^{i,j}$ and $\ell_3^{i,j}$. To simplify notation we will write, for any function f, $f_{i,j} = f(x_i, y_j)$.
Using the Simpson rule in x and the trapezoidal rule in y we obtain,

$$\int_{R_{ij}} \frac{\partial e}{\partial x} v \, dx \, dy = \frac{h_i h_j}{12} \left\{ \left(\frac{\partial e}{\partial x} v\right)_{i,j-1} + \left(\frac{\partial e}{\partial x} v\right)_{i,j} + \left(\frac{\partial e}{\partial x} v\right)_{i-1,j-1} + \left(\frac{\partial e}{\partial x} v\right)_{i-1,j} \right\}.$$

But, using again that $\frac{\partial e}{\partial x}$ vanishes at the midpoints of $\ell_1^{i,j}$ and $\ell_3^{i,j}$, we have

$$\left(\frac{\partial e}{\partial x}\right)_{i,j-1} = \left(\frac{\partial e}{\partial x}\right)_{i,j} = -\left(\frac{\partial e}{\partial x}\right)_{i-1,j-1} = -\left(\frac{\partial e}{\partial x}\right)_{i-1,j} = \frac{h_i}{2} \frac{\partial^2 e}{\partial x^2}$$

and therefore,

$$\begin{aligned} \int_{R_{ij}} \frac{\partial e}{\partial x} v \, dx \, dy &= \frac{h_i^2}{12} \frac{\partial^2 e}{\partial x^2} \left\{ \frac{h_j}{2} (v_{i,j-1} + v_{i,j}) - \frac{h_j}{2} (v_{i-1,j-1} + v_{i-1,j}) \right\} \\ &= \frac{h_i^2}{12} \left(\int_{\ell_2^{i,j}} \frac{\partial^2 p}{\partial x^2} v \, dy - \int_{\ell_4^{i,j}} \frac{\partial^2 p}{\partial x^2} v \, dy \right) \end{aligned}$$

as we wanted to show. $\square$

In the next lemma we will use a standard trace theorem. For $w \in H^1(R_{ij})$,

$$\|w\|_{0,\ell_r^{ij}} \leq C \left\{ h_i^{-1/2} \|w\|_{0,R_{ij}} + h_i^{1/2} \left\| \frac{\partial w}{\partial x} \right\|_{0,R_{ij}} \right\} \tag{3.7}$$

with $r = 2, 4$. When $v \in \mathcal{Q}_1(R_{ij})$, the second term on the right hand side can be bounded by the first one using an inverse inequality. Therefore, in that case we have,

$$\|v\|_{0,\ell_r^{ij}} \leq C \left\{ h_i^{-1/2} \|v\|_{0,R_{ij}} \right\} \tag{3.8}$$

Lemma 3.3. *Let u be the solution of* (1.1). *There exists a constant C such that, for any $v \in V_h$,*

$$\left| \int_\Omega b \cdot \nabla(u - u_I) v \, dx \, dy \right| \leq C h^2 \log^3(1/\varepsilon) \, \|v\|_\varepsilon$$

Proof. Let Pb be the piecewise constant approximation of b defined by $Pb|_{R_{ij}} := b^{i,j}$, where $b^{i,j}$ denotes the value of b at the barycenter of R_{ij}. We have

$$\int_\Omega b \cdot \nabla(u - u_I) v \, dx \, dy = \int_\Omega (b - Pb) \cdot \nabla(u - u_I) v \, dx \, dy + \int_\Omega Pb \cdot \nabla(u - u_I) v \, dx \, dy \tag{3.9}$$

Let us estimate each term on the right-hand side. Since the derivatives of b are bounded we have $\|b - Pb\|_\infty \leq Ch|b|_{1,\infty}$ and so, for each element R_{ij},

$$\left| \int_{R_{ij}} (b - Pb) \cdot \nabla(u - u_I) v \, dx \, dy \right| \leq Ch \, \|\nabla(u - u_I)\|_{0,R_{ij}} \, \|v\|_{0,R_{ij}} \, .$$

Therefore, summing over all indices i,j such that $R_{ij} \subset \Omega_1$, it follows that

$$\left| \int_{\Omega_1} (b - Pb) \cdot \nabla(u - u_I) v \, dx \, dy \right| \le Ch \left\| \nabla(u - u_I) \right\|_{0,\Omega_1} \left\| v \right\|_{0,\Omega_1} . \tag{3.10}$$

On the other hand, since v vanishes at the boundary of Ω, it follows from Poincaré inequality that

$$\|v\|_{0,\Omega_1} \le C\varepsilon \log(1/\varepsilon) \left\| \frac{\partial v}{\partial x} \right\|_{0,\Omega_1} \tag{3.11}$$

and therefore, using the estimate

$$\varepsilon^{1/2} \left\| \nabla(u - u_I) \right\|_{0,\Omega_1} \le Ch,$$

which was proved in [5, Theorem 2.1], we obtain from (3.10),

$$\left| \int_{\Omega_1} (b - Pb) \cdot \nabla(u - u_I) v \, dx \, dy \right| \le Ch^2 \log(1/\varepsilon) \left\| v \right\|_{\varepsilon} .$$

Clearly, the same argument can be applied to obtain an analogous estimate over Ω_2. Finally, for $R_{ij} \in \Omega_3$, we use (2.5) and a standard interpolation error estimate to obtain

$$\left| \int_{\Omega_3} (b - Pb) \cdot \nabla(u - u_I) v \, dx \, dy \right| \le Ch^2 \left\| v \right\|_{0,\Omega_3} .$$

Summing up we conclude that

$$\left| \int_{\Omega} (b - Pb) \cdot \nabla(u - u_I) v \, dx \, dy \right| \le Ch^2 \log(1/\varepsilon) \left\| v \right\|_{\varepsilon} . \tag{3.12}$$

Now we estimate the second term on the right hand side of (3.9). We have

$$\int_{\Omega} Pb \cdot \nabla(u - u_I) v \, dx \, dy = \sum_{i,j=1}^{M} \int_{R_{ij}} b_1^{i,j} \frac{\partial(u - u_I)}{\partial x} v \, dx \, dy + \sum_{i,j=1}^{M} \int_{R_{ij}} b_2^{i,j} \frac{\partial(u - u_I)}{\partial y} v \, dx \, dy$$

and we will estimate the first term on the right hand side (clearly the second one can be handled in an analogous way). From the definition of K_{ij} (3.6) it follows that

$$\begin{aligned} &\sum_{i,j=1}^{M} \int_{R_{ij}} b_1^{i,j} \frac{\partial(u - u_I)}{\partial x} v \, dx \, dy \\ &\quad = \sum_{i,j=1}^{M} b_1^{i,j} K_{ij}(u,v) + \sum_{i,j=1}^{M} \frac{b_1^{i,j} h_i^2}{12} \left(\int_{\ell_2^{i,j}} \frac{\partial^2 u}{\partial x^2} v \, dy - \int_{\ell_4^{i,j}} \frac{\partial^2 u}{\partial x^2} v \, dy \right) . \end{aligned} \tag{3.13}$$

Then, it is enough to bound the right hand side of (3.13). For the first term we write,

$$K_{ij}(u,v) = K_{1,ij}(u,v) - K_{2,ij}(u,v)$$

with

$$K_{1,ij}(u,v) = \int_{R_{ij}} \frac{\partial(u - u_I)}{\partial x} v \, dx \, dy$$

and

$$K_{2,ij}(u,v) = \frac{h_i^2}{12} \left(\int_{\ell_2^{i,j}} \frac{\partial^2 u}{\partial x^2} v \, dy - \int_{\ell_4^{i,j}} \frac{\partial^2 u}{\partial x^2} v \, dy \right) .$$

From Lemma 3.2 we know that, for any $p \in \mathcal{P}_2(R_{ij})$,

$$K_{ij}(u,v) = K_{ij}(u-p,v) = K_{1,ij}(u-p,v) - K_{2,ij}(u-p,v).$$

Now, taking $p \in \mathcal{P}_2(R_{ij})$ satisfying (3.2) and (3.3) and using the interpolation error estimate (3.1) for $w = u - p$ we obtain,

$$|K_{1,ij}(u-p,v)| \leq C \left\{ h_i^2 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}} + h_i h_j \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}} + h_j^2 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}} \right\} \|v\|_{0,R_{ij}}.$$

On the other hand, using now (3.7) for $w = \frac{\partial^2 u}{\partial x^2}$, (3.8), and again (3.2), we get

$$|K_{2,ij}(u-p,v)| \leq C \left\{ h_i^2 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}} + h_i h_j \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}} \right\} \|v\|_{0,R_{ij}}.$$

In conclusion we have,

$$|K_{ij}(u,v)| \leq C \left\{ h_i^2 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}} + h_i h_j \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}} + h_j^2 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}} \right\} \|v\|_{0,R_{ij}}. \tag{3.14}$$

Now we are ready to estimate the first term on the right hand side of (3.13). Setting

$$I_s := \sum_{i,j:R_{ij} \subset \Omega_s} b_1^{i,j} K_{ij}(u,v), \qquad s = 1,2,3,$$

we have

$$\sum_{i,j=1}^{M} b_1^{i,j} K_{ij}(u,v) = I_1 + I_2 + I_3.$$

From (3.14), using the Cauchy-Schwarz inequality we obtain,

$$|I_1| \leq C \left\{ \sum_{i,j:R_{ij} \subset \Omega_1} \left(h_i^4 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}}^2 + h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}}^2 + h_j^4 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}}^2 \right) \right\}^{\frac{1}{2}} \|v\|_{0,\Omega_1}$$

and therefore, using now the Poincaré inequality (3.11),

$$|I_1| \leq C \log(1/\varepsilon) \left\{ \sum_{i,j:R_{ij} \subset \Omega_1} \left(\varepsilon h_i^4 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}}^2 + \varepsilon h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}}^2 + \varepsilon h_j^4 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}}^2 \right) \right\}^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \left\| \frac{\partial v}{\partial x} \right\|_{0,\Omega_1}.$$

Now, for $R_{i1} \subset \Omega_1$ we have $h_i \leq c_1 \varepsilon h \log(1/\varepsilon)$ and $h_1 = \varepsilon h$, then

$$\begin{aligned} &\varepsilon h_i^4 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{i1}}^2 + \varepsilon h_i^2 h_1^2 \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{i1}}^2 + \varepsilon h_1^4 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{i1}}^2 \\ &\qquad \leq C h^4 \log^4(1/\varepsilon) \left(\varepsilon^5 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{i1}}^2 + \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{i1}}^2 + \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{i1}}^2 \right). \end{aligned}$$

If $R_{ij} \subset \Omega_1$, with $j \geq 2$, we use that $h_i \leq c_1 \varepsilon h \log(1/\varepsilon)$ and that $h_j \leq hy$ for all $(x,y) \in R_{ij}$, obtaining

$$\begin{aligned} &\varepsilon h_i^4 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}}^2 + \varepsilon h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}}^2 + \varepsilon h_j^4 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}}^2 \\ &\qquad \leq C h^4 \log^4(1/\varepsilon) \left(\varepsilon^5 \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}}^2 + \varepsilon^3 \left\| y \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,R_{ij}}^2 + \varepsilon \left\| y^2 \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{0,R_{ij}}^2 \right). \end{aligned}$$

Therefore,

$$\begin{aligned}|I_1| \quad &\leq \quad Ch^2\log^2(1/\varepsilon)\left\{\varepsilon^5\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\Omega_1}^2+\varepsilon^5\left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,\Omega_1}^2\right.\\&\left.+\varepsilon^5\left\|\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,\Omega_1}^2+\varepsilon^3\left\|y\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,\Omega_1}^2+\varepsilon\left\|y^2\frac{\partial^3 u}{\partial x\partial y^2}\right\|_{0,\Omega_1}^2\right\}^{1/2}\varepsilon^{\frac12}\left\|\frac{\partial v}{\partial x}\right\|_{0,\Omega_1},\end{aligned}$$

and consequently, using the weighted estimates (2.8), (2.9), and (2.10), we obtain

$$|I_1| \leq Ch^2\log^3(1/\varepsilon)\,\|v\|_\varepsilon\,.$$

An analogous argument can be used to estimate I_2. Finally, for I_3, using (2.5) and (3.14), we arrive at

$$|I_3| \leq Ch^2\,\|v\|_{0,\Omega_3}\,.$$

Therefore, we conclude that

$$\left|\sum_{i,j=1}^{M} b_1^{i,j}K_{ij}(u,v)\right| \leq Ch^2\log^3(1/\varepsilon)\,\|v\|_\varepsilon\,.$$

To finish the proof it remains only to estimate the second term in (3.13). Observe that, for $1\leq i\leq M-1$, we have $\ell_2^{i,j}=\ell_4^{i+1,j}$ and $\int_{\ell_4^{1,j}}\frac{\partial^2 u}{\partial x^2}v\,dy=\int_{\ell_2^{M,j}}\frac{\partial^2 u}{\partial x^2}v\,dy=0$. Therefore,

$$\begin{aligned}&\sum_{i,j=1}^{M}\frac{b_1^{i,j}h_i^2}{12}\left(\int_{\ell_2^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy-\int_{\ell_4^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\right)\\&\quad=\sum_{j=1}^{M}\left\{\frac{b_1^{1,j}h_1^2}{12}\int_{\ell_2^{1,j}}\frac{\partial^2 u}{\partial x^2}v\,dy+\sum_{i=2}^{M-1}\frac{b_1^{i,j}h_i^2}{12}\left(\int_{\ell_2^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy-\int_{\ell_4^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\right)-\frac{b_1^{M,j}h_M^2}{12}\int_{\ell_4^{M,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\right\}\\&\quad=\sum_{j=1}^{M}\sum_{i=1}^{M-1}\left(\frac{b_1^{i,j}h_i^2-b_1^{i+1,j}h_{i+1}^2}{12}\right)\int_{\ell_2^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\end{aligned}\tag{3.15}$$

From the definition of the mesh, we know that $h_1>h_2$, so for $i=1$ we have:

$$\begin{aligned}\left|(b_1^{1,j}h_1^2-b_1^{2,j}h_2^2)\int_{\ell_2^{1,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\right| &\leq Ch_1^2\left|\int_{\ell_2^{1,j}}\int_0^x\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}v\right)(t,y)dt\,dy\right|\\&\leq Ch_1^2\int_{R_{1j}}\left|\frac{\partial^3 u}{\partial x^3}v+\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x}\right|dx\,dy\\&\leq Ch_1^2\left(\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{1j}}\|v\|_{0,R_{1j}}+\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{1j}}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{1j}}\right)\end{aligned}$$

We call $\widetilde\Omega_1=\bigcup_{j=1}^M R_{1j}$. Since $h_1=\varepsilon h$, we obtain

$$\begin{aligned}\left|\sum_{j=1}^{M}(b_1^{1,j}h_1^2-b_1^{2,j}h_2^2)\int_{\ell_2^{1,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\right| &\leq Ch_1^2\sum_{j=1}^{M}\left(\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{1j}}\|v\|_{0,R_{1j}}+\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{1j}}\left\|\frac{\partial v}{\partial x}\right\|_{0,R_{1j}}\right)\\&\leq Ch^2\left(\varepsilon^2\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\widetilde\Omega_1}\|v\|_{0,\widetilde\Omega_1}+\varepsilon^{3/2}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,\widetilde\Omega_1}\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,\widetilde\Omega_1}\right)\end{aligned}$$

Since v vanishes at the boundary of Ω, it follows from Poincaré inequality that

$$\|v\|_{0,\widetilde{\Omega}_1} \leq Ch\varepsilon \left\| \frac{\partial v}{\partial x} \right\|_{0,\widetilde{\Omega}_1} \tag{3.16}$$

and then

$$\begin{aligned}\left| \sum_{j=1}^{M} (b_1^{1,j} h_1^2 - b_1^{2,j} h_2^2) \int_{\ell_2^{1,j}} \frac{\partial^2 u}{\partial x^2} v \, dy \right| &\leq Ch^2 \left(\varepsilon^{5/2} \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,\widetilde{\Omega}_1} + \varepsilon^{3/2} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{0,\widetilde{\Omega}_1} \right) \varepsilon^{1/2} \left\| \frac{\partial v}{\partial x} \right\|_{0,\widetilde{\Omega}_1} \\ &\leq Ch^2 \|v\|_\varepsilon\end{aligned}$$

For $i \geq 2$, it follows from the definition of the mesh (and the assumption $h < 1$) that $h_{i+1}^2 - h_i^2 = h^2 h_i (x_{i-1} + x_i)$, $h_{i+1} \leq 2h_i$, and $x_{i-1} + x_i \leq 3x_{i-1}$, then

$$\left| b_1^{i,j} h_i^2 - b_1^{i+1,j} h_{i+1}^2 \right| \leq Ch^2 h_i x_{i-1},$$

and

$$\begin{aligned}\left| \sum_{j=1}^{M} \sum_{i=2}^{M-1} \left(\frac{b_1^{i,j} h_i^2 - b_1^{i+1,j} h_{i+1}^2}{12} \right) \int_{\ell_2^{i,j}} \frac{\partial^2 u}{\partial x^2} v \, dy \right| &\leq C \sum_{j=1}^{M} \sum_{i=2}^{M-1} h^2 h_i x_{i-1} \left| \int_{\ell_2^{i,j}} \frac{\partial^2 u}{\partial x^2} v \right| dy \\ &\leq C \sum_{j=1}^{M} \sum_{i=2}^{M-1} h^2 h_i x_{i-1} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{0,\ell_2^{i,j}} \|v\|_{0,\ell_2^{i,j}}\end{aligned}$$

Using the inequalities (3.7) and (3.8) we obtain,

$$\begin{aligned}&\left| \sum_{j=1}^{M} \sum_{i=2}^{M-1} \left(\frac{b_1^{i,j} h_i^2 - b_1^{i+1,j} h_{i+1}^2}{12} \right) \int_{\ell_2^{i,j}} \frac{\partial^2 u}{\partial x^2} v \, dy \right| \\ &\quad \leq Ch^2 \sum_{j=1}^{M} \sum_{i=2}^{M-1} h_i x_{i-1} \left(h_i^{-1/2} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{0,R_{ij}} + h_i^{1/2} \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}} \right) h_i^{-1/2} \|v\|_{0,R_{ij}} \\ &\quad \leq Ch^2 \sum_{j=1}^{M} \sum_{i=2}^{M-1} x_{i-1} \left(\left\| \frac{\partial^2 u}{\partial x^2} \right\|_{0,R_{ij}} + h_i \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,R_{ij}} \right) \|v\|_{0,R_{ij}} .\end{aligned}$$

Now, for $R_{ij} \subset \Omega_1 \backslash \widetilde{\Omega}_1$, using the Cauchy-Schwarz inequality, the Poincaré inequality (3.11) and the weighted inequalities (2.6) and (2.8), we have

$$
\begin{aligned}
&\sum_{R_{ij}\subset\Omega_1\setminus\widetilde{\Omega}_1} x_{i-1}\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{ij}} + h_i\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}}\right)\|v\|_{0,R_{ij}} \\
&\quad\le C\left\{\sum_{R_{ij}\subset\Omega_1\setminus\widetilde{\Omega}_1} x_{i-1}^2\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{ij}} + h_i\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}}\right)^2\right\}^{1/2}\|v\|_{0,\Omega_1} \\
&\quad\le C\left\{\sum_{R_{ij}\subset\Omega_1\setminus\widetilde{\Omega}_1} (c_1\varepsilon\log(1/\varepsilon))^2\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{ij}} + \varepsilon\log(1/\varepsilon)\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}}\right)^2\right\}^{1/2}\varepsilon\log(1/\varepsilon)\left\|\frac{\partial v}{\partial x}\right\|_{0,\Omega_1} \\
&\quad\le C\log^3(1/\varepsilon)\left\{\sum_{R_{ij}\subset\Omega_1\setminus\widetilde{\Omega}_1}\left(\varepsilon^{3/2}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{ij}} + \varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}}\right)^2\right\}^{1/2}\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,\Omega_1} \\
&\quad\le C\log^3(1/\varepsilon)\left(\varepsilon^{3/2}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,\Omega_1} + \varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\Omega_1}\right)\varepsilon^{1/2}\left\|\frac{\partial v}{\partial x}\right\|_{0,\Omega_1} \\
&\quad\le C\log^3(1/\varepsilon)\,\|v\|_\varepsilon\,.
\end{aligned}
$$

For $\Omega_2\cup\Omega_3$, using (2.5) we have

$$
\sum_{R_{ij}\subset\Omega_2\cup\Omega_3} x_{i-1}\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,R_{ij}} + h_i\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,R_{ij}}\right)\|v\|_{0,R_{ij}} \le C\,\|v\|_\varepsilon\,.
$$

Collecting all the estimates we obtain

$$
\sum_{i=1}^M\sum_{j=1}^M b_1^{i,j}h_i^2\left(\int_{\ell_2^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy - \int_{\ell_4^{i,j}}\frac{\partial^2 u}{\partial x^2}v\,dy\right) \le Ch^2\log^3(1/\varepsilon)\,\|v\|_\varepsilon
$$

concluding the proof of the lemma. □

In the next lemma we give an estimate for the reaction term of the equation. This estimate follows immediately from results in [5].

Lemma 3.4. *Let u be the solution of* (1.1). *There exists a constant C such that, for any $v\in V_h$,*

$$
\left|\int_\Omega c(u-u_I)v\,dx\,dy\right| \le Ch^2\,\|v\|_\varepsilon
$$

Proof. From [5, Theorem 2.1] we know that $\|u-u_I\|_{0,\Omega}\le Ch^2$, hence

$$
\begin{aligned}
\left|\int_\Omega c(u-u_I)v\,dx\,dy\right| &\le C\,\|u-u_I\|_{0,\Omega}\,\|v\|_{0,\Omega} \\
&\le Ch^2\,\|v\|_{0,\Omega} \le Ch^2\,\|v\|_\varepsilon\,.
\end{aligned}
$$

□

We can now state and prove our main result which says that, the ε-norm of the difference between the interpolation of the exact solution u_I and the finite element approximation u_h, is of higher order than the ε-norm of the error $u-u_h$.

Theorem 3.5. *Let u be the solution of* (1.1)*, $u_h \in V_h$ its finite element approximation and $u_I \in V_h$ its Lagrange interpolation. There exists a constant C such that,*

$$\|u_h - u_I\|_\varepsilon \le Ch^2 \log^3(1/\varepsilon)$$

Proof. From (2.3) and the error equation $\mathcal{B}(u - u_h, u_h - u_I) = 0$, we have

$$\beta \|u_h - u_I\|_\varepsilon^2 \le \mathcal{B}(u_h - u_I, u_h - u_I) = \mathcal{B}(u - u_I, u_h - u_I).$$

But, from Lemmas 3.1, 3.3 and 3.4, we have

$$\mathcal{B}(u - u_I, u_h - u_I) \le C \log^3(1/\varepsilon) h^2 \|u_h - u_I\|_\varepsilon ,$$

and therefore the theorem is proved. □

An immediate consequence of the theorem combined with the interpolation results proved in [5] is the optimal order convergence in the L^2-norm.

Corollary 3.6. *Let u be the solution of* (1.1) *and $u_h \in V_h$ its finite element approximation. There exists a constant C such that,*

$$\|u - u_h\|_{0,\Omega} \le C \log^3(1/\varepsilon) h^2$$

Proof. The result follows immediately from the interpolation error estimate $\|u - u_I\|_{0,\Omega} \le Ch^2$ proved in [5, Theorem 2.1] and the estimate given in Theorem 3.5. □

We end this section giving error estimates in terms of the number of nodes.

Corollary 3.7. *Let u be the solution of* (1.1) *and $u_h \in V_h$ its finite element approximation. If N is the number of nodes in $\mathcal{T}_h$ then, there exists a constant C such that,*

$$\|u_h - u_I\|_\varepsilon \le C \frac{\log^5(1/\varepsilon)}{N}$$

and

$$\|u - u_h\|_{0,\Omega} \le C \frac{\log^5(1/\varepsilon)}{N}.$$

Proof. The results follow from Theorem 3.5, Corollary 3.6 and the estimate

$$h \le C \frac{\log(1/\varepsilon)}{\sqrt{N}}.$$

which was proved in [5, Corollary 2.3]. □

4 A higher order approximation by postprocessing

As it is known, superconvergence results of the type of Theorem 3.5 can be used to improve the numerical approximation by some local postprocessing. In this section we construct a higher order approximation u_h^* of the solution of (1.1), obtained from the computed finite element approximation $u_h \in V_h$ by a simple local procedure. We follow the approach given in [6].
For simplicity we consider now the meshes defined as indicated in Remark 2.1. In this way the lengths in each direction of neighbor elements are comparable and this simplifies the analysis.
We define the postprocessed solution u_h^* as in [6, 14]. We repeat the construction given in those papers for the sake of completeness. Assume that the mesh $\mathcal{T}_h$ is a refinement of a coarser mesh formed by elements S_{ij} which are as in Figure 2 (note that we assume that $\mathcal{T}_h$ contains an even number of elements). We define $u_h^* = I_2 u_h$, where $I_2 u_h$ is the biquadratic interpolation of u_h on S_{ij}, over the nine nodes indicated in Figure 2, i.e., the vertices of the elements of the original mesh.
We want to show that u_h^* is a higher order approximation in the ε-norm. We will need the following estimates for the biquadratic interpolation.

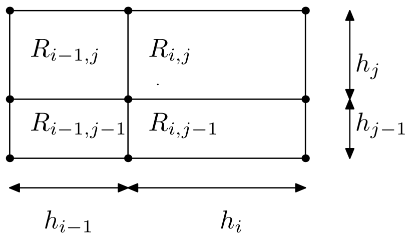


Figure 2: Reference element for the Q_2-interpolation and region S_{ij}

Lemma 4.1. *Let u be the solution of* (1.1) *and I_2u the piecewise biquadratic interpolation of u on the mesh made with the elements S_{ij} and using the nodes corresponding to the vertices of the original mesh (as indicated in Figure 2). There exists a constant C such that*

$$\|u - I_2u\|_\varepsilon \leq Ch^2 \tag{4.1}$$

Proof. The inequality is an easy consequence of the a priori estimates given in Lemma 2.2 and the following error estimates for the interpolation operator I_2. Let H_i and H_j be the lengths of the element S_{ij} along the directions of the x and y axis respectively. Then, for $v \in H^3(S_{ij})$, we have

$$\|v - I_2v\|_{L^2(S_{ij})} \leq C \left\{ H_i^3 \left\| \frac{\partial^3 v}{\partial x^3} \right\|_{L^2(S_{ij})} + H_j^3 \left\| \frac{\partial^3 v}{\partial y^3} \right\|_{L^2(S_{ij})} \right\}, \tag{4.2}$$

$$\left\| \frac{\partial(v - I_2v)}{\partial x} \right\|_{L^2(S_{ij})} \leq C \left\{ H_i^2 \left\| \frac{\partial^3 v}{\partial x^3} \right\|_{L^2(S_{ij})} + H_j^2 \left\| \frac{\partial^3 v}{\partial x \partial y^2} \right\|_{L^2(S_{ij})} \right\}, \tag{4.3}$$

and

$$\left\| \frac{\partial(v - I_2v)}{\partial y} \right\|_{L^2(S_{ij})} \leq C \left\{ H_i^2 \left\| \frac{\partial^3 v}{\partial x^2 \partial y} \right\|_{L^2(S_{ij})} + H_j^2 \left\| \frac{\partial^3 v}{\partial y^3} \right\|_{L^2(S_{ij})} \right\} \tag{4.4}$$

where the constant C is independent of the element S_{ij} and v. For the standard biquadratic interpolation, these inequalities are proved in [1, Theorem 2.7]. The only difference between our case and that considered in [1], is that we are not using the usual interpolation nodes. Indeed, our interpolation nodes on S_{ij} are (ξ_k, ξ_l), with $k = i-2, i-1, i$ and $l = j-2, j-1, j$, i.e., we have moved a little bit the nodes usually located at edge mid-points and barycenter of the elements. However, it follows from the definition of the meshes $\mathcal{T}_h$ that the ratios $(\xi_i - \xi_{i-1})/(\xi_{i-1} - \xi_{i-2})$ are uniformly bounded from below and above. Using this fact, it is not difficult to see that the arguments used in [1] can be adapted to our case.
Let us now prove (4.1). Since $H_1 \leq Ch\varepsilon$, using (4.3) for the element S_{11} we obtain

$$\left\| \frac{\partial(u - I_2u)}{\partial x} \right\|^2_{L^2(S_{11})} \leq Ch^4 \left\{ \varepsilon^4 \left\| \frac{\partial^3 u}{\partial x^3} \right\|^2_{L^2(S_{11})} + \varepsilon^4 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|^2_{L^2(S_{11})} \right\}.$$

Now, for $S_{i1}, i > 1$, using now $H_1 \leq Ch\varepsilon$ and $H_i \leq Chx$ for all $(x, y) \in S_{i1}$, we have

$$\left\| \frac{\partial(u - I_2u)}{\partial x} \right\|^2_{L^2(S_{i1})} \leq Ch^4 \left\{ \left\| x^2 \frac{\partial^3 u}{\partial x^3} \right\|^2_{L^2(S_{i1})} + \varepsilon^4 \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|^2_{L^2(S_{i1})} \right\}.$$

Analogously, for $S_{1j}, j > 1$, we use that $H_1 \leq Ch\varepsilon$ and $K_j \leq Chy$ for all $(x, y) \in S_{1j}$ to obtain

$$\left\| \frac{\partial(u - I_2u)}{\partial x} \right\|^2_{L^2(S_{1j})} \leq Ch^4 \left\{ \varepsilon^4 \left\| \frac{\partial^3 u}{\partial x^3} \right\|^2_{L^2(S_{1j})} + \left\| y^2 \frac{\partial^3 u}{\partial x \partial y^2} \right\|^2_{L^2(S_{1j})} \right\}.$$

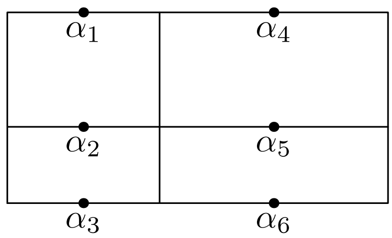


Figure 3: Interpolation points for Q_{12} on S_{ij}

Finally, for $i, j > 1$, we have $H_i \le Chx$ and $H_j \le Chy$ for all $(x, y) \in S_{ij}$, and so,

$$\left\| \frac{\partial(u - I_2 u)}{\partial x} \right\|^2_{L^2(S_{ij})} \le Ch^4 \left\{ \left\| x^2 \frac{\partial^3 u}{\partial x^3} \right\|^2_{L^2(S_{ij})} + \left\| y^2 \frac{\partial^3 u}{\partial x \partial y^2} \right\|^2_{L^2(S_{ij})} \right\}.$$

Therefore, multiplying by ε, summing up, and using the a priori estimates (2.8) and (2.9), we obtain

$$\varepsilon^{\frac{1}{2}} \left\| \frac{\partial(u - I_2 u)}{\partial x} \right\|_{L^2(\Omega)} \le Ch^2.$$

In a similar way, using now (4.4) and (4.2), we can prove

$$\varepsilon^{\frac{1}{2}} \left\| \frac{\partial(u - I_2 u)}{\partial y} \right\|_{L^2(\Omega)} \le Ch^2 \qquad \text{and} \qquad \|v - I_2 v\|_{L^2(\Omega)} \le Ch^2.$$

Therefore, (4.1) holds. $\square$

Lemma 4.2. *There exists a constant C such that, for any $v \in V_h$,*

$$\|I_2 v\|_\varepsilon \le C \|v\|_\varepsilon \tag{4.5}$$

Proof. It is easy to see that the Lagrange basis functions corresponding to I_2 are bounded independently of h. Indeed, this follows from the fact that the ratios h_i/h_{i-1} are uniformly bounded. Consequently we have

$$\|I_2 v\|_{L^\infty(S_{ij})} \le C \|v\|_{L^\infty(S_{ij})}.$$

Therefore, using the Schwarz inequality and the inverse inequality

$$\|v\|_{L^\infty(S_{ij})} \le \frac{C}{|S_{ij}|^{\frac{1}{2}}} \|v\|_{0,S_{ij}},$$

we obtain

$$\|I_2 v\|_{0,S_{ij}} \le C \|v\|_{0,S_{ij}}.$$

On the other hand, for $v \in V_h$, $\frac{\partial(I_2 v)}{\partial x}$ can be seen as a Lagrange type interpolation of $\frac{\partial v}{\partial x}$. Indeed, $\frac{\partial(I_2 v)}{\partial x}$ is the unique polynomial in Q_{12} (the space of polynomials of degree one in the x variable and two in the y variable) such that

$$\frac{\partial v}{\partial x}(\alpha_j) = \frac{\partial(I_2 v)}{\partial x}(\alpha_j), \qquad j = 1, \cdots, 6$$

where the points α_j are those indicated in Figure 3. Then, using again that h_i/h_{i-1} are uniformly bounded and an inverse inequality, we obtain

$$\left\| \frac{\partial(I_2 v)}{\partial x} \right\|_{0,S_{ij}} \le C \left\| \frac{\partial v}{\partial x} \right\|_{0,S_{ij}}.$$

Analogously, we can prove

$$\left\|\frac{\partial(I_2 v)}{\partial y}\right\|_{0,S_{ij}} \leq C\left\|\frac{\partial v}{\partial y}\right\|_{0,S_{ij}}$$

and so, the lemma is proved. $\square$

We can now give the main result of this section.

Theorem 4.3. *Let u be the solution of* (1.1)*, $u_h \in V_h$ its finite element approximation and $u_h^* = I_2 u_h$. There exists a constant C such that,*

$$\|u - u_h^*\|_\varepsilon \leq C \log^3(1/\varepsilon) h^2.$$

Proof. Since $I_2 u_I = I_2 u$, we have

$$\|u - u_h^*\|_\varepsilon \leq \|u - I_2 u\|_\varepsilon + \|I_2(u_I - u_h)\|_\varepsilon$$

and therefore, using (4.1), (4.5) and Theorem 3.5, we conclude the proof. $\square$

5 Numerical Experiments

We end the paper with some numerical results. We consider problem (1.1) with

$$b = (1-2\varepsilon)(-1,-1)\ , \quad c = 2(1-\varepsilon),$$

and the right hand side given by

$$f(x,y) = -\left[x - \left(\frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right) + y - \left(\frac{1-e^{-\frac{y}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x+y}.$$

In this case the exact solution is

$$u(x,y) = \left[\left(x - \frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\left(y - \frac{1-e^{-\frac{y}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x+y},$$

In Tables 1 and 2 we present the results for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-6}$ respectively. Recall that N denotes the number of nodes.

N	h	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_\varepsilon$	$\|u_I-u_h\|_\varepsilon$	$\|u-u_h^*\|_\varepsilon$
625	0.375000	0.011851	0.142881	0.020495	0.040664
841	0.320000	0.007778	0.121252	0.012387	0.027535
1089	0.275000	0.006511	0.104864	0.009772	0.021133
1681	0.215000	0.004717	0.082466	0.006485	0.013545
2601	0.170000	0.002924	0.065163	0.003068	0.007807
3969	0.135000	0.002113	0.051935	0.002022	0.005047
5929	0.110000	0.001349	0.042269	0.001863	0.003686
16129	0.065000	0.000538	0.025071	0.000995	0.001581
22201	0.055000	0.000400	0.021228	0.000661	0.001089

Table 1: $\varepsilon = 10^{-3}$

N	h	$\|u - u_h\|_{L^2}$	$\|u - u_h\|_\varepsilon$	$\|u_I - u_h\|_\varepsilon$	$\|u - u_h^*\|_\varepsilon$
2025	0.390000	0.013595	0.148975	0.023704	0.045450
2601	0.340000	0.009248	0.128921	0.015445	0.032214
3249	0.300000	0.006961	0.113833	0.010700	0.024212
5625	0.220000	0.004565	0.084096	0.006050	0.013562
10201	0.160000	0.002444	0.061215	0.002998	0.007048
16129	0.125000	0.001728	0.047970	0.002055	0.004523
20449	0.110000	0.001407	0.042271	0.001372	0.003378
29929	0.090000	0.001031	0.034640	0.001019	0.002348
37249	0.080000	0.000903	0.030837	0.000694	0.001795

Table 2: $\varepsilon = 10^{-6}$.

With these numerical results we have computed the order of the different errors in terms of N. The computed orders, for the case $\varepsilon = 10^{-3}$, are shown in Figure 4. The picture in the left shows the order of the errors $\|u - u_h\|_{L^2}$ and $\|u - u_h\|_\varepsilon$, and that in the right the different errors in the ε-norm.
Observe that, the order of $\|u - u_h\|_{L^2}$ is -0.9104, which essentially agrees with that predicted by the theory which is -1. Similarly, the orders shown in the second picture agree with the theoretical ones.

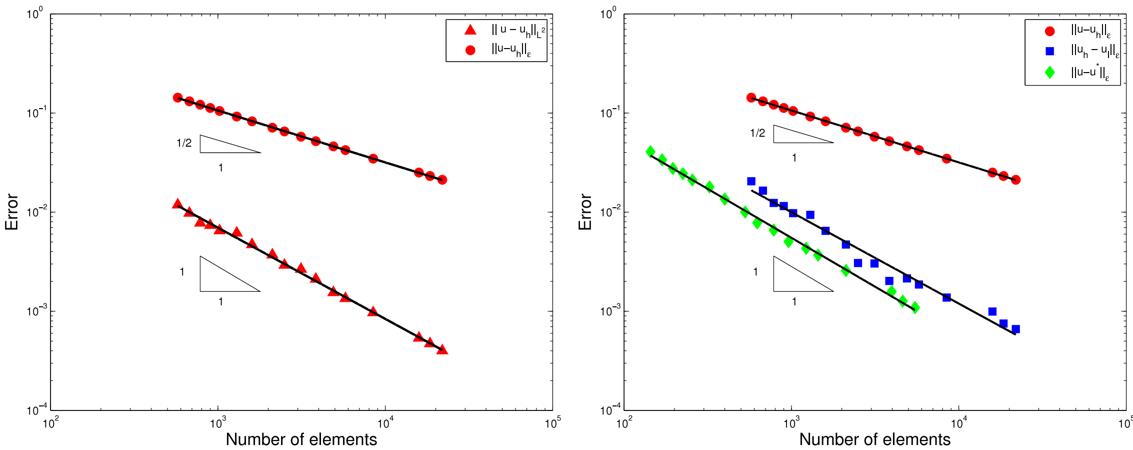
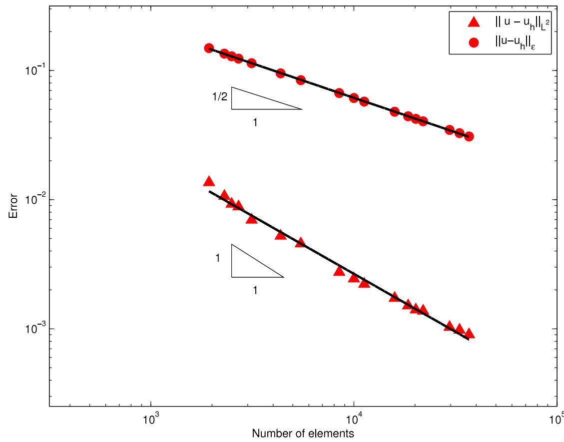


Figure 4: Numerical orders, $\varepsilon = 10^{-3}$.

Next we show analogous pictures for the case $\varepsilon = 10^{-6}$ in Figure 5. Again, the estimated orders agree with those given by the theory.



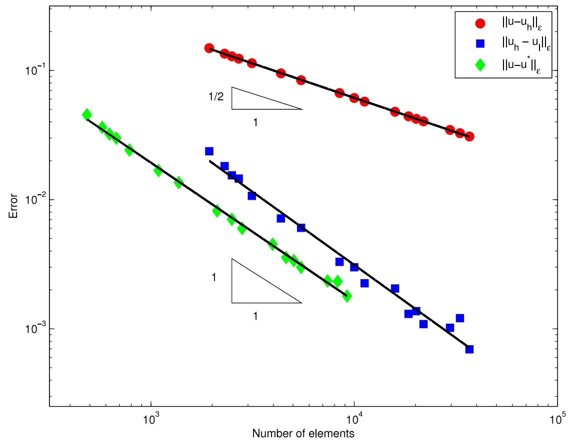


Figure 5: Numerical orders, $\varepsilon = 10^{-6}$.

As mentioned in the introduction, an advantage of the graded meshes over the Shishkin type meshes is that, for the first ones, the meshes generated for some value of ε work well also for larger values. This fact was observed in [5], where numerical results comparing the errors with both kind of meshes were presented. Our numerical experiments show that a similar behavior is observed for superconvergence. In Table 3 we give the values of $\|u_I - u_h\|_\varepsilon$ for several values of ε using both kind of meshes with the same number of nodes (11236 elements) generated for the case $\varepsilon = 10^{-6}$.

ε	Graded mesh	Shishkin mesh
10^{-1}	0.003109	0.002021
10^{-2}	0.002438	0.194888
10^{-3}	0.002107	1.010498
10^{-4}	0.002049	0.972038
10^{-5}	0.002042	0.390427
10^{-6}	0.002254	0.002250

Table 3: $\|u_I - u_h\|_\varepsilon$ for both kind of meshes.

Finally, using a sequence of graded meshes designed for $\varepsilon = 10^{-6}$, with different number of nodes, we have computed the order in terms of $1/N$ of the different errors for the computed solutions and its postprocessed ones, for several values of $\varepsilon \geq 10^{-6}$. The results are shown in Table 4. As it is seen, the superconvergence order for $\|u_I - u_h\|_\varepsilon$ ans $\|u - u_h^*\|_\varepsilon$ is obtained for all the values of ε considered.

ε	$\|u-u_h\|_\varepsilon$	$\|u_I-u_h\|_\varepsilon$	$\|u-u_h^*\|_\varepsilon$
10^{-1}	0.50683	0.99225	1.0294
10^{-2}	0.51433	0.89524	1.0021
10^{-3}	0.51409	0.96446	1.0294
10^{-4}	0.51397	0.99008	1.039
10^{-5}	0.51426	0.9934	1.0403
10^{-6}	0.54049	1.0694	1.0724

Table 4: Errors for different values of ε with meshes generated for $\varepsilon = 10^{-6}$.

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