A MIXED DISCRETIZATION OF ELLIPTIC PROBLEMS ON POLYHEDRA USING ANISOTROPIC HYBRID MESHES

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Abstract. A Virtual Element Method is introduced for the mixed approximation of a simple model problem for the Laplace operator on a polyhedron. The method is fully analysed when the meshes are made up of triangular right prisms, pyramids and tetrahedra. The local discrete spaces coincide with the lowest order Raviart-Thomas spaces on tetrahedral and triangular right prismatic elements, and extend them to pyramidal elements. The discrete scheme is well posed and optimal interpolation error estimates are proved on meshes which allow for anisotropic elements. In particular, local interpolation error estimates for the discrete element space are optimal and anisotropic on anisotropic right prisms. Furthermore, a discretization of the model problem in the presence of edge and vertex singularities is analysed for the proposed method on a family of suitably designed graded meshes, and optimal estimates for the approximation error are obtained, extending in this way the results of [Farhloul, Nicaise, Paquet, ESAIM: M2AN 35 (2001) 907–920] where cylindrical domains with edge singularities were considered.

1. Introduction

This paper is mainly motivated by the following observation regarding the approximation properties of the Raviart-Thomas space $RT(T)$ on a tetrahedron T. There exists a constant $C(\bar{c})$ depending only on $\bar{c}$ such that if T satisfies the *regular vertex property* with parameter $\bar{c}$ (RVP($\bar{c}$)) and $\mathbf{u} \in H^1(T)^3$ then

$$\|\mathbf{u} - \Pi_0 \mathbf{u}\|_{L^2(T)} \leq C(\bar{c}) \left(\sum_{i=1}^{3} h_i \|\partial_{x_i} \mathbf{u}\|_{L^2(T)} + h_T \|\operatorname{div} \mathbf{u}\|_{L^2(T)} \right), \tag{1}$$

where Π_0 is the Raviart-Thomas interpolation operator of lowest order [17, 21], h_T is the diameter of T and h_i is the diameter of T in the x_i-direction. The tetrahedron T satisfies RVP($\bar{c}$) if it has a vertex $\mathtt{v}$ such that if ℓ_i, $i = 1, 2, 3$ are the versors with directions of the edges sharing $\mathtt{v}$ and M is the matrix made up column-wise with ℓ_i then $\det M > \bar{c}$. The result is also valid for higher order Raviart-Thomas interpolation [1]. A less restrictive geometrical property is the *maximum angle condition* (MAC). We say that a tetrahedron T satisfies MAC($\bar{\alpha}$) if the angles of the faces of T and between faces are less than $\bar{\alpha}$. We know that there exists a constant $C(\bar{\alpha})$ depending only on $\bar{\alpha}$ such that for all T which satisfies MAC($\bar{\alpha}$) it holds

$$\|\mathbf{u} - \Pi_0 \mathbf{u}\|_{L^2(T)} \leq C(\bar{\alpha}) h_T |\mathbf{u}|_{H^1(T)}, \tag{2}$$

for all $\mathbf{u} \in H^1(T)^3$. Inequality (2) is weaker than (1), since there are elements satisfying MAC($\bar{\alpha}$) for a fixed $\bar{\alpha}$ with arbitrarily small RVP parameter $\bar{c}$, making the constant $C(\bar{c})$ in (1) to degenerate [1]. Furthermore, inequality (1) cannot be proved under the maximum angle condition as stated in [1] by means of a counterexample.

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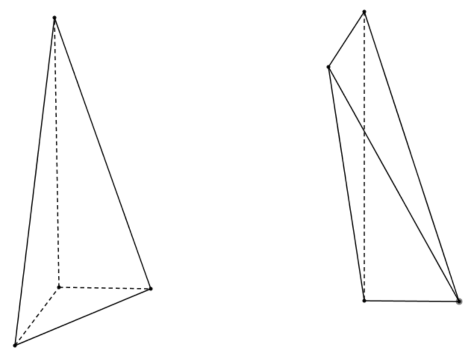

FIGURE 1. Tetrahedra satisfying MAC($\frac{\pi}{2}$). The tetrahedron at the right satisfies RVP with a poor constant close to 0.

In several situations in mixed finite element approximations the use of meshes with narrow elements is needed. This is the case for instance when dealing with the Poisson equation

$$\begin{aligned} -\Delta p &= f \qquad \text{on } \Omega \\ p &= 0 \qquad \text{in } \partial\Omega, \end{aligned} \tag{3}$$

in a polyhedron Ω with concave edges, which, introducing the vectorial variable $\mathbf{u} = -\nabla p$ can be written as (3) is

$$\left\{ \begin{aligned} \mathbf{u} &= -\nabla p \qquad \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= f \qquad \text{in } \Omega \\ p &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \right. \tag{4}$$

In this case, $\mathbf{u}$ is in general not in H^1 due to vertex and edges singularities. In particular, close to concave edges, the solution is expected to be more regular in its direction than transversally to it, and consequently the mesh has to be accordingly refined in order to recover optimal order of convergence with respect to the number of degrees of freedom [2, 3, 4]. Those meshes contain elements which are arbitrarily elongated in the direction of concave edges (direction in which the solution is more regular). It is possible to construct this kind of meshes with tetrahedral elements all satisfying MAC($\bar{\alpha}$) for $\bar{\alpha} < \pi$ fixed, but unfortunately those elements do not satisfy RVP with a parameter uniformly far from 0. That is due to the presence of tetrahedra with bounded maximal angle but poor regular vertex constant, see Figure 1. So, interpolation error estimate (1) cannot be globally used to estimate the error approximation, but (2) has to be taken, and consequently, the anisotropic properties of the meshes may give no benefit.

An idea to overcome this difficulty, for the case of Ω being a cylindrical polyhedral domain, was proposed in [13]. In this case, when f is in $L^2(\Omega)$, the solution may exhibit only singularities along concave edges. Then the authors proposed a lowest order mixed Raviart–Thomas method on graded anisotropic meshes made up of triangular right prisms and they proved optimal error estimates by means of adequate anisotropic interpolation results. In this way, tetrahedra which do not satisfy a uniform regular vertex property are avoided.

Interestingly, also in [13], and again for cylindrical domains, a mixed Raviart–Thomas method on the tetrahedral anisotropic graded mesh which is obtained by splitting the prismatic elements into three tetrahedra. Of course, these kind of meshes contains the bad elements which are avoided with the prismatic ones. However, in order to obtain optimal approximation error estimates, the price to be paid is to require additional regularity on the right hand side, precisely, it has to belong to a weighted Sobolev space.

In this paper we extend the result of [13] in order to be able to deal with the mixed approximation of (3) with $f \in L^2(\Omega)$ in general polyhedral domains for which the solution may exhibit singularities of both edge and vertex types. Since such domains cannot be always meshed by means of right prisms and we also would like to avoid to require more regularity to f as mentioned in the previous paragraph, we propose a discretization based on hybrid meshes. Similarly to [13] for primatic meshes, with the use of our proposed hybrid meshes, we obtain the optimal estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}. \tag{5}$$

with $\mathbf{u}_h$ and p_h being the approximations of the solutions $\mathbf{u}$ and p of (4). To do that, firstly we introduce and analyze a new finite/virtual element space V_h in $\mathbb{R}^3$ with the following conditions:

- C.1 Conformity: The space V_h must be $H(\mathrm{div})$-conforming.
- C.2 Optimal and anisotropic approximation properties: Optimal interpolation error estimates have to be valid even on families of meshes which do not satisfy the standard shape-regularity condition [11].
- C.3 Domain generality: The space has to be well defined on conforming (without hanging nodes) polyhedral meshes without restricting the considered domains to few special polyhedra.

As suggested in [10] for the 2d case, we present V_h as a virtual element space on a conforming polyhedral mesh, which locally coincides with the original lowest order 3d Raviart-Thomas space on tetrahedra and right prisms, and extends it naturally to pyramidal elements. In particular, normal components of the discrete functions are constants on the faces of the elements, fitting well across different shape's elements. In this way requirement C.1 is verified. An advantage of this presentation is that the definition of the local spaces is independent of the geometry of the element (see Section 2). Also, in [7] an analogous space, but of arbitrary order, was introduced to discretize an acoustic flow free vibration problem in a bounded rigid cavity in $\mathbb{R}^2$.

The virtual element method (VEM) has been recently introduced [5] as a generalization of H^1–conforming finite elements to arbitrary element-geometry and as a generalization of Mimetic Finite Differences to arbitrary degree of accuracy and arbitrary continuity properties. An extension to the discretization of $H(\mathrm{div})$–conforming vector fields and mixed finite element approximations has been proposed in [10] in the two dimensional case. Furthermore, in [6] a mixed VEM has been analysed for the approximation of general linear elliptic problems with variable coefficients. The virtual element space can contain non piecewise polynomial functions, and mainly, functions which are a priori unknown, in the sense that they cannot be explicitly evaluated. In the VEM approach, the space and the degrees of freedom are taken in such a way that the elementary stiffness matrix can be computed without actually computing these non-polynomial functions, but just using the degrees of freedom. In this respect, a key point in this approach is that, given an element E, if $\mathbf{u} = \nabla q_2$ for a known (quadratic, in this paper) polynomial q_2, then for a field $\mathbf{v}$ the quantity

$$\int_E \mathbf{u} \cdot \mathbf{v}$$

can be computed if $\mathrm{div}\,\mathbf{v}$ and the outer normal component $\mathbf{v} \cdot \mathbf{n}$ of $\mathbf{v}$ are known polynomials (constants in our case) on E and ∂E respectively, since

$$\begin{aligned}\int_E \mathbf{u} \cdot \mathbf{v} &= \int_E \nabla q_2 \cdot \mathbf{v} \\ &= -\int_E q_2 \mathrm{div}\,\mathbf{v} + \int_{\partial E} q_2 \mathbf{v} \cdot \mathbf{n}.\end{aligned}$$

In order to satisfy condition C.2 and taking into account what we remarked at the beginning of this Section, we need to avoid the use of a kind of anisotropic tetrahedra. Following [13] we allow for arbitrarily anisotropic (triangular) right prisms for which optimal anisotropic local interpolation error are proved. But the use of only right prisms would restrict too much the domains which can be considered, and because of that we further allow for tetrahedral elements, and pyramids (of parallelogram basis) in order to glue right prisms and tetrahedra together. Our interpolation error estimates depend on the aspect ratio of the pyramids (and, for simplicity, also of tetrahedra), and so we are implicitly imposing that this kind of elements must be uniformly isotropic. However we do not lose generality, since meshes adapted to general singularities in polyhedra can be constructed, as we show at the end of the article, satisfying this requirement. Then conditions C.2 and C.3 are also satisfied.

Hybrid meshes including tetrahedral and prismatic (and even hexahedral) elements may be needed to satisfy the demands of a specific problem geometry (complex regions) or to reach efficient calculations. If these meshes are to avoid hanging nodes then they will in general contain pyramids, see for instance [19]. Several authors have introduced and analysed conforming finite elements on pyramids, some of them are [8] for H^1–elements, [14, 18] for $H(\mathrm{div})$– and $H(\mathbf{curl})$–elements, the first one for lowest order and the second one for higher order. In [18] it is proved that it is not possible to construct useful H^1-finite elements on pyramids using only polynomial functions. In the $H(\mathrm{div})$ case, it is explained also in the same article that all the spaces constructed in the literature contain non–polynomial functions.

For the sake of simplicity , we just consider the simplest problem of the mixed formulation (3) with $f \in L^2(\Omega)$ and Ω a polyhedral domain. Its mixed variational formulation can be written as: to find $\mathbf{u} \in V$ and $p \in Q$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) &= 0 \qquad \forall \mathbf{v} \in V \\ b(\mathbf{u},q) &= (f,q) \qquad \forall q \in Q, \end{aligned} \tag{6}$$

with

$$a(\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{v}\cdot\mathbf{w}, \qquad b(\mathbf{v},q) = \int_\Omega q \,\mathrm{div}\,\mathbf{v}$$

and

$$V = H(\mathrm{div}\,,\Omega), \qquad Q = L^2(\Omega).$$

Of course, the problem for the $\mathrm{div}(a\nabla)$ operator can be similarly treated.

We remark that meshes with more general polyhedral-shaped elements can be considered. Indeed that is one of the VEM's main features. But we decided to restrict ourselves to few (but without loss of generality) shapes since our main objective is to allow for meshes with anisotropic elements, and with uniformly valid anisotropic estimates. This is fulfilled by allowing right prisms, since the local space for those elements becomes known (it was introduced in [17]) and that allows to obtain stability and interpolation error estimates. One difficulty when other shapes are considered (like oblique prisms, for instance) is that the local VEM space is not preserved by Piola transformation (the vanishing **curl** property is not preserved), and so standard rescaling arguments are hard to use.

The outline of the paper is as follows. In Section 2 the discrete spaces for the discrete mixed formulation are defined. Then, in Section 3, the variational discrete forms are stated. The discrete form, a_h, requires of projections on subspaces of the local discrete fields. Those projections become the identity in case of tetrahedral or prismatic elements, but for pyramids, they require some analysis which is performed in Section 4. In Section 5 the interpolation on the virtual element spaces is considered and interpolation error estimates are proved under different shape assumptions. Also a discrete inf-sup property is proved, which is used in Section 6 to give an abstract approximation error result. In Section 7, approximation error estimates on general polyhedral domains when the solution has vertices and edges singularities are given using specially designed hybrid meshes which are constructed and analyzed in Section 8.

We use the standard notation for Lebesgue and Sobolev spaces, norms and seminorms for functions and fields, and $H(\text{div}, S)$ (resp. $H(\mathbf{curl}, S)$) denote the spaces of $L^2(S)^3$ fields with divergence (resp. $\mathbf{curl}$) in $L^2(S)$ (resp. $L^2(S)^3$). Fields will be denoted by lower case bold face letters such as $\mathbf{u}, \mathbf{v}$, and $\mathbf{x} = (x_1, x_2, x_3)$ will denote the variable in $\mathbb{R}^3$. Given a field $\mathbf{v}$, its components are v_i, $i = 1, 2, 3$, that is $\mathbf{v} = (v_1, v_2, v_3)$. The space of polynomials of degree less than or equal k is denoted by $\mathcal{P}_k$. Given a mesh $\mathcal{T}$ of a domain Ω, $\mathcal{P}_k(\mathcal{T})$ denotes the space of functions on Ω whose restriction to each element of $\mathcal{T}$ is in $\mathcal{P}_k$. P_0^S denotes the $L^2(S)$-projection on $\mathcal{P}_0$ and $P_0^{\mathcal{T}}$ denotes the $L^2(\Omega)$ projection on $\mathcal{P}_0(\mathcal{T})$. We denote by h_D the diameter of the set $D \subset \mathbb{R}^n$, $n = 1, 2, 3$. The letters c or C denote constants which may depend on parameters which are specified in the text, and they may vary from one place to another. With $a \sim b$ we mean that $a \leq Cb$ and $b \leq Ca$ hold (similarly, we use the symbol $\lesssim$).

We finish this introduction with a brief discussion and some definitions, following [4], about the geometric singularities of the solutions $\mathbf{u}$ and p of (6) when the domain Ω is a general polyhedron.

Let S be a corner of Ω. Let C_S be the infinite polyhedral cone that coincides with Ω in a neighborhood of S. Define $G_S = C_S \cap \mathcal{S}^2(S)$, where $\mathcal{S}^2(S)$ is the unit sphere centered at S. Then, the vertex singular exponent related to S is given by $\lambda_{v,S} = -\frac{1}{2} + \sqrt{\lambda_{S,1} + \frac{1}{4}}$, where $\lambda_{S,k} > 0, k = 1, \ldots$, are the eigenvalues, in increasing order, of the Laplace-Beltrami operator on G_S with Dirichlet boundary conditions. Note that $\lambda_{v,S} > 0$. We say that the vertex S is singular if $\lambda_{v,S} < \frac{1}{2}$.

Now, let A be an edge of Ω. The edge singular exponent related to A is $\lambda_{e,A} = \pi/\omega_A$, with ω_A being the angle between the two faces containing A. Note that $\lambda_{e,A} > \frac{1}{2}$. We say that A is singular if $\lambda_{e,A} < 1$.

If a vertex or edge is not singular, we say that it is regular.

It follows that we can decompose the set $\mathcal{C}$ of the corners of Ω into two disjoint subsets $\mathcal{C}_s$ and $\mathcal{C}_r$ containing the singular and regular corners, respectively. A similar decomposition $\mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_r$ is done for the set $\mathcal{E}$ of edges of Ω.

Assuming a decomposition of $\Omega = \cup_{\ell=1}^N \Lambda_\ell$ in tetrahedral macroelements having at most a singular edge and a singular vertex, we have the following regularity result. First we introduce the space $V_{\beta,\delta}^{1,2}(\Lambda)$ for a macroelement Λ as

$$V_{\beta,\delta}^{1,2}(\Lambda) = \left\{ v \in \mathcal{D}'(\Lambda) : R^{\beta-1+|\alpha|} \theta^{\delta-1+|\alpha|} D^\alpha v \in L^2(\Lambda), \alpha \in \mathbb{N}_0^3, |\alpha| \leq 1 \right\}$$

where $R(\mathbf{x})$ is the distance of $\mathbf{x}$ to the vertices of Λ, $r(\mathbf{x})$ is the distance from $\mathbf{x}$ to the edges of Λ and finally $\theta(\mathbf{x})$ is the angular distance $\theta(\mathbf{x}) = \frac{r(\mathbf{x})}{R(\mathbf{x})}$.

Theorem 1.1. *The solutions $\mathbf{u}$ and p of problem (6) satisfy*

$$p \in H^1(\Omega)$$

and for each ℓ

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_s$$

with $\mathbf{u}_r \in H^1(\Omega)$ and

$$\mathbf{u}_s \cdot \xi_i \in V_{\beta,\delta}^{1,2}(\Lambda_\ell), \quad i = 1, 2, \qquad \mathbf{u}_s \cdot \xi_3 \in V_{\beta,0}^{1,2}(\Lambda_\ell)$$

where ξ_i, $i = 1, 2, 3$, are the directions of three concurrent edges of Λ_ℓ with ξ_3 being the direction of the singular edge if it exists in Ω_ℓ, and $\beta, \delta \geq 0$ satisfying $\beta > \frac{1}{2} - \lambda_v^{(\ell)}$ and $\delta > 1 - \lambda_e^{(\ell)}$, v and e being the singular vertex and edge, respectively, if they exist.

2. The discrete spaces

Given a polyhedral triangulation $\mathcal{T}_h$ of Ω, we introduce the finite element spaces V_h and Q_h in order to approximate problem (6).

On one hand, we are interested in allowing for hybrid meshes, that is, meshes containing polyhedral elements of different shapes. On the other hand, we want to allow for arbitrarily anisotropic elements, and so, the global shape-regularity property of the mesh [11, Chapter 3] cannot be assumed. In order to

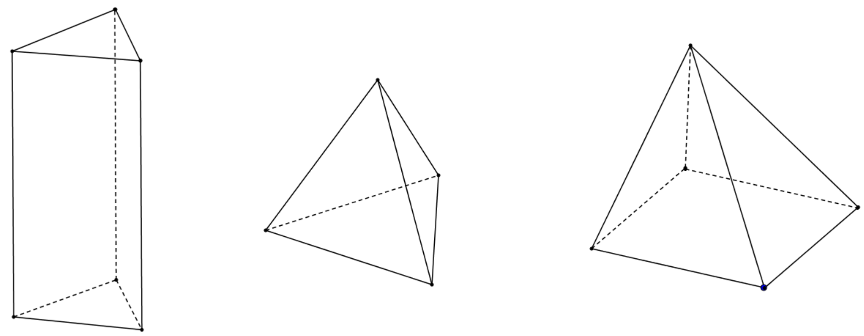

FIGURE 2. Meshes may contain triangular right prisms, tetrahedra or pyramids with parallelogram basis.

clarify the exposition, we introduce some of the geometric assumptions on the meshes that we will need in this and the next Sections. We assume that:

- G1 For each $h > 0$, the mesh $\mathcal{T}_h$ of Ω is conforming and made up of tetrahedra, triangular right prisms and pyramids with parallelogram basis, see Figure 2. Only these kinds of elements are considered in this paper. We assume that the diameters of the elements are less than h.
- G2 The aspect ratios of every tetrahedron and pyramid in $\mathcal{T}_h$, for $h > 0$, are uniformly bounded by a constant σ_r independent of h.

We will add a third condition in Section 6. Recall that the aspect ratio of an element E is the quotient $\frac{h_E}{\rho_E}$ between the diameter h_E of E and the diameter ρ_E of the largest ball contained in $\bar{E}$.

For each element $E \in \mathcal{T}_h$ we define the local space

$$V_h(E) \quad = \quad \Big\{ \mathbf{v} \in H(\mathrm{div}, E) \cap H(\mathbf{curl}, E) : \ \mathbf{v} \cdot \mathbf{n} \in \mathcal{P}_0(f) \ \forall f \text{ face of } E, \ \mathrm{div}\, \mathbf{v} \in \mathcal{P}_0(E), \ \mathbf{curl}\, \mathbf{v} = 0 \Big\},$$

with $\mathbf{n}$ being the outer normal to ∂E. This definition was suggested in [10, Section 5] in the 2d case. A similar space, in the 2d case, but for arbitrary order was also used in [7]. Then we define the global space

$$V_h \quad = \quad \{ \mathbf{v} \in H(\mathrm{div}, \Omega) : \mathbf{v}|_E \in V_h(E), \forall E \in \mathcal{T}_h \} \, .$$

We consider the following degrees of freedom

$$\int_f \mathbf{v} \cdot \mathbf{n}, \qquad f \text{ face of } E, \quad E \in \mathcal{T}_h \tag{7}$$

For use in later Sections we define here the scalar discrete space as

$$Q_h = \mathcal{P}_0(\mathcal{T}_h). \tag{8}$$

Lemma 2.1. *Given a polyhedron E, the degrees of freedom (restricted to E) define a unique $\mathbf{v}$ in $V_h(E)$.*

Proof. We need to construct a function $\mathbf{v}$ with given degrees of freedom (7). Let g be the function on ∂E, constant on each face f of E, such that

$$\int_f g = \int_f \mathbf{v} \cdot \mathbf{n}, \qquad f \text{ face of } E,$$

and let $d \in \mathcal{P}_0(E)$ such that

$$\int_E d = \int_{\partial E} \mathbf{v} \cdot \mathbf{n}.$$

Then we consider the problem

$$\Delta\phi = d \quad \text{in } E, \qquad \frac{\partial\phi}{\partial\mathbf{n}} = g \quad \text{on } \partial E, \qquad \int_E \phi = 0.$$

This problem has a unique solution since the compatibility condition

$$\int_E d = \int_{\partial E} g$$

is fulfilled. We define $\mathbf{v} = \nabla\phi$. Then $\operatorname{div}\mathbf{v} = d \in \mathcal{P}_0$, $\mathbf{curl}\,\mathbf{v} = 0$ and $\mathbf{v}\cdot\mathbf{n} = g \in \mathcal{P}_0(f)$ for all $f \subseteq \partial E$. So $\mathbf{v} \in V_h(E)$.

On the other hand, suppose that $\mathbf{v} \in V_h(E)$ has vanishing degrees of freedom. Since $\mathbf{curl}\,\mathbf{v} = 0$ it follows that $\mathbf{v} = \nabla\phi$ for some function $\phi \in H^1(\Omega)$. Now, since $\operatorname{div}\mathbf{v} \in \mathcal{P}_0(E)$

$$0 = \int_{\partial E} \mathbf{v}\cdot n = \int_E \operatorname{div}\mathbf{v}$$

implies $\operatorname{div}\mathbf{v} = 0$. So, ϕ satisfies

$$\Delta\phi = 0, \quad \text{in } \Omega, \qquad \frac{\partial\phi}{\partial\mathbf{n}} = 0, \quad \text{on } \partial E$$

which implies ϕ is constant. So $\mathbf{v} = 0$. □

If $\mathbf{v} \in [W^{1,1}(E)]^3$ we define the $V_h(E)$-interpolation $\mathbf{v}_I \in V_h(E)$ as the function in $V_h(E)$ such that

$$\int_f \mathbf{v}_I\cdot\mathbf{n} = \int_f \mathbf{v}\cdot\mathbf{n}, \qquad \forall f \text{ face of } E.$$

It follows from Lemma 2.1 that actually there exists such a $\mathbf{v}_I$. We easily have

$$\operatorname{div}\mathbf{v}_I = P_0^E \operatorname{div}\mathbf{v}, \qquad \text{on E,} \tag{9}$$

where P_0^E is the $L^2(E)$-projection onto the space of constant functions on E.

Proposition 2.2. *If E is a tetrahedron then*

$$V_h(E) = \{\mathbf{v} = (a + \gamma x_1, b + \gamma x_2, c + \gamma x_3) : a, b, c, \gamma \in \mathbb{R}\},$$

with x_1, x_2, x_3 being the variables in a Cartesian system of coordinates. If E is a triangular right prism then

$$V_h(E) = \{\mathbf{v} = (a + \gamma x_1, b + \gamma x_2, c + d x_3) : a, b, c, d, \gamma \in \mathbb{R}\},$$

with x_1, x_2, x_3 being the variables in a Cartesian system of coordinates with the x_3-axis perpendicular to the planes containing the triangular basis of E.

Proof. We can check that properties defining $V_h(E)$ are satisfied, recalling that the div and **curl** operators can be computed in the local variables x_1, x_2, x_3 of the statement. This proves that, in each case, the proposed spaces are contained in $V_h(E)$. Furthermore $\dim V_h(E)$ is 4 in the case of tetrahedra and 5 in the case of prisms, which coincide with the dimension of the proposed spaces. □

Remark 2.3. *The spaces $V_h(E)$ in the previous Proposition coincide with the $H(div)$ conforming spaces of lowest order introduced in* [17] *as a generalization of Raviart-Thomas spaces to tetrahedra and prisms.*

Remark 2.4. *As a consequence of the previous Proposition, we see that the Raviart-Thomas spaces of lowest order on tetrahedra and right prisms are independent of the choice of the Cartesian axes, whenever the x_3-axis is perpendicular to the triangular basis in case of prisms. This can be proved directly. For* *instance, in the case of prisms, let $x_1x_2x_3$ and $x_1'x_2'x_3'$ be two Cartesian coordinates systems satisfying the required properties. Then we have*

$$\begin{aligned} x_1 &= p + \alpha x_1' - \beta x_2' \\ x_2 &= q + \beta x_1' + \alpha x_2' \\ x_3 &= r + x_3' \end{aligned}$$

with $\alpha^2 + \beta^2 = 1$. Let $\mathbf{v}(x_1, x_2, x_3) = (a + \gamma x_1, b + \gamma x_2, c + dx_3)$. So,

$$\mathbf{v}(x_1, x_2, x_3) = [(a + \gamma p) + \gamma(\alpha x_1' - \beta x_2'), (b + \gamma q) + \gamma(\beta x_1' + \alpha x_2'), (c + dr) + dx_3'] .$$

Then, the components of $\mathbf{v}$ in the new coordinate versors are

$$\begin{aligned} \mathbf{v} \cdot (\alpha, \beta, 0) &= (\alpha a + \beta b + \gamma(\alpha p + \beta q)) + \gamma x_1' =: a' + \gamma x_1' \\ \mathbf{v} \cdot (-\beta, \alpha, 0) &= (-\beta a + \alpha b + \gamma(-\beta p + \alpha q)) + \gamma x_2' =: b' + \gamma x_2' \\ \mathbf{v} \cdot (0, 0, 1) &= (c + dr) + dx_3' =: c' + dx_3'. \end{aligned}$$

It follows that in the $x_1'x_2'x_3'$ system of coordinates, $\mathbf{v}(x_1', x_2', x_3') = (a' + \gamma x_1', b' + \gamma x_2', c' + dx_3')$.

3. The discrete problem

In this Section we follow closely the general lines developed in [10]. We consider the decomposition

$$a(\mathbf{v}, \mathbf{w}) = \sum_{E \in \mathcal{T}_h} a^E(\mathbf{v}, \mathbf{w}), \qquad b(\mathbf{v}, q) = \sum_{E \in \mathcal{T}_h} b^E(\mathbf{v}, q).$$

Since for all $q \in Q_h$ we have

$$b(\mathbf{v}, q) = \int_\Omega q \operatorname{div} \mathbf{v} = \sum_{E \in \mathcal{T}_h} \int_E q \operatorname{div} \mathbf{v} = \sum_{E \in \mathcal{T}_h} \int_{\partial E} q \mathbf{v} \cdot \mathbf{n},$$

we note that for $(\mathbf{v}, q) \in V_h \times Q_h$, $b(\mathbf{v}, q)$ can be computed using the degrees of freedom (7) applied to $\mathbf{v}$.

Now we introduce for each element E the space

$$W(E) = \{\mathbf{v} \in V_h(E) : \mathbf{v} = \nabla q_2, \text{ for some } q_2 \in \mathcal{P}_2(E)\} .$$

If $\mathbf{v} \in V_h(E), \mathbf{u} = \nabla q_2 \in W(E)$, then

$$\begin{aligned} a^E(\mathbf{u}, \mathbf{v}) &= \int_E \mathbf{u} \cdot \mathbf{v} = \int_E \nabla q_2 \cdot \mathbf{v} \\ &= -\int_E q_2 \operatorname{div} \mathbf{v} + \int_{\partial E} q_2 \mathbf{v} \cdot \mathbf{n} \end{aligned}$$

which can be computed from the degrees of freedom of $\mathbf{v}$. Now, for $\mathbf{v} \in V_h$ we introduce the projection $\Pi_w^E \mathbf{v} \in W(E)$ defined by

$$a^E(\mathbf{v} - \Pi_w^E \mathbf{v}, \mathbf{w}) = 0 \qquad \forall \mathbf{w} \in W(E).$$

We note that $\Pi_w^E \mathbf{v}$ can also be computed from the degrees of freedom of $\mathbf{v}$.

Now we define

$$a_h^E(\mathbf{v}, \mathbf{w}) = a^E(\Pi_w^E \mathbf{v}, \Pi_w^E \mathbf{w}) + h_E^{-1} \mathcal{S}^E((I - \Pi_w^E)\mathbf{v}, (I - \Pi_w^E)\mathbf{w}), \qquad \mathbf{v}, \mathbf{w} \in V_h(E),$$

with $\mathcal{S}^E$ the bilinear form on $V_h(E) \times V_h(E)$ associated with the identity matrix in $\mathbb{R}^{n^E}$ with respect to the local dual basis of the degrees of freedom (7).

Remark 3.1. *If E is a tetrahedron or a right prism, then $W(E)$ coincides with $V_h(E)$. Therefore, in that case, $\Pi_w^E \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V_h(E)$ and hence $a_h^E(\mathbf{v}, \mathbf{w}) = a^E(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V_h(E)$.*

Finally we introduce $a_h : V_h \times V_h \to \mathbb{R}$ by

$$a_h(\mathbf{v}, \mathbf{w}) = \sum_{E \in \mathcal{T}_h} a_h^E(\mathbf{v}, \mathbf{w}).$$

In what follows we analyse the second term in a_h^E when E is a pyramid. In this case, denote by $\{\mathbf{v}_i\}$ the dual basis of the degrees of freedom (7), that is,

$$\mathbf{v}_i \in V_h(E), \qquad \text{and} \qquad \int_{f_j} \mathbf{v}_i \cdot \mathbf{n} = \delta_{i,j}, \quad 1 \le i, j \le 5,$$

if f_i, $i = 1, \dots, 5$, denote the faces of E. Then $\mathbf{v} \in V_h(E)$ can be uniquely written as

$$\mathbf{v} = \sum_{i=1}^{5} a_i \mathbf{v}_i,$$

with

$$a_i = \int_{f_i} \mathbf{v} \cdot \mathbf{n}.$$

By setting

$$|||\mathbf{v}|||_E^2 = \frac{1}{h_E} \sum_{i=1}^{5} a_i^2,$$

$|||\cdot|||_E$ defines a norm on $V_h(E)$, and since it has a finite dimension, we know that for constants c_E and C_E, which depend on E, we have

$$c_E \|\mathbf{v}\|_{L^2(E)} \le |||\mathbf{v}|||_E \le C_E \|\mathbf{v}\|_{L^2(E)}, \qquad \forall \mathbf{v} \in V_h(E). \tag{10}$$

The purpose of the next Proposition is to prove that C_E and c_E can be taken depending only on the aspect ratio of E.

Proposition 3.2. *Let E be a pyramid, and consider the basis $\{\mathbf{v}_i, i = 1, \dots, 5\}$ of $V_h(E)$, and the associated discrete norm $|||\cdot|||_E$ introduced above. Then there exist constants C_E and c_E depending only on the aspect ratio of E such that* (10) *holds true for all $\mathbf{v} \in V_h$.*

Proof. First we note that if $\mathbf{v} \in V_h(E)$ is given by

$$\mathbf{v} = \sum_{i=1}^{5} a_i \mathbf{v}_i$$

then it satisfies

$$\mathbf{v} = \nabla \phi$$

with

$$\begin{aligned} \Delta \phi &= d && \text{in } E \\ \frac{\partial \phi}{\partial \mathbf{n}} &= g && \text{on } \partial E \\ \int_E \phi &= 0 \end{aligned} \tag{11}$$

for

$$g|_{f_i} = \frac{a_i}{|f_i|}, \quad 1 \le i \le 5, \qquad |E| d = \sum_{i=1}^{5} a_i. \tag{12}$$

Given $q \in H^1(E)$, we multiply the first equation of (11) by q and integrate on E, integrate by parts on the left hand side, and use the Neumann boundary conditions to have

$$\int_E \nabla\phi \cdot \nabla q = -\int_E dq + \int_{\partial E} gq.$$

Since

$$\|\mathbf{v}\|_{L^2(E)} = \|\nabla\phi\|_{L^2(E)} = \sup\left\{\int_E \nabla\phi \cdot \nabla q : \; q \in H^1(E), \|\nabla q\|_{L^2(E)} = 1, \int_E q = 0\right\}$$

we conclude that

$$\begin{aligned} \|\mathbf{v}\|_{L^2(E)} &= \sup\left\{-\int_E dq + \int_{\partial E} gq : \; q \in H^1(E), \|\nabla q\|_{L^2(E)} = 1, \int_E q = 0\right\} \\ &= \sup\left\{\int_{\partial E} gq : \; q \in H^1(E), \|\nabla q\|_{L^2(E)} = 1, \int_E q = 0\right\} \end{aligned} \tag{13}$$

(using, in the last equality, that d is constant on E and q has vanishing integral there). Since d and g can be written in terms of a_i, we have obtained an expression of the norm $\|\mathbf{v}\|_{L^2(E)}$ in terms of the coefficients of $\mathbf{v}$ in the basis. Now, let $\hat{E}$ be a reference pyramid, and $F : \hat{E} \to E$ an affine transformation mapping $\hat{E}$ onto E (remember that only pyramids with parallelogram basis are considered), which can be written as

$$\mathbf{x} = F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}} + \mathbf{c}.$$

Given $q \in H^1(E)$ we define $\hat{q}$ by

$$\hat{q}(\hat{\mathbf{x}}) = q(\mathbf{x}), \qquad \forall \hat{\mathbf{x}} \in \hat{E}$$

and we observe that there exists constants c_0 and c_1 depending on the aspect ratio of E such that

$$\frac{c_0}{h_E}\|\nabla q\|^2_{L^2(E)} \le \|\nabla \hat{q}\|^2_{L^2(\hat{E})} \le \frac{c_1}{h_E}\|\nabla q\|^2_{L^2(E)}, \tag{14}$$

and, on the other hand,

$$\int_{\hat{E}} \hat{q} = 0 \quad \Longleftrightarrow \quad \int_E q = 0.$$

We have

$$\int_{\partial E} gq = \|\nabla\hat{q}\|_{L^2(\hat{E})}\left(\int_{\partial\hat{E}} \hat{g}\frac{\hat{q}}{\|\nabla\hat{q}\|_{L^2(\hat{E})}}|J|\right)$$

with $|J(\hat{\mathbf{x}})|_{f_i}| = |f_i|/|\hat{f}_i|$. It follows that

$$\|\mathbf{v}\|_{L^2(E)} = \sup\left\{\|\nabla\hat{q}\|_{L^2(\hat{E})}\int_{\partial\hat{E}} \hat{g}|J|\frac{\hat{q}}{\|\nabla\hat{q}\|_{L^2(\hat{E})}} : \quad q \in H^1(E), \|\nabla q\|_{L^2(E)} = 1, \int_E q = 0 \quad\right\},$$

and taking (14) into account we obtain

$$\|\mathbf{v}\|_{L^2(E)} \le \frac{c_1^{\frac{1}{2}}}{h_E^{\frac{1}{2}}}\sup\left\{\int_{\partial\hat{E}} \hat{g}|J|\hat{q} : \quad \hat{q} \in H^1(\hat{E}), \|\nabla\hat{q}\|_{L^2(\hat{E})} = 1, \int_{\hat{E}} \hat{q} = 0 \quad\right\} \tag{15}$$

and

$$\|\mathbf{v}\|_{L^2(E)} \ge \frac{c_0^{\frac{1}{2}}}{h_E^{\frac{1}{2}}}\sup\left\{\int_{\partial\hat{E}} \hat{g}|J|\hat{q} : \quad \hat{q} \in H^1(\hat{E}), \|\nabla\hat{q}\|_{L^2(\hat{E})} = 1, \int_{\hat{E}} \hat{q} = 0 \quad\right\}. \tag{16}$$

We remark that

$$\int_{\hat{E}} d|B| = \int_{\partial\hat{E}} \hat{g}|J|,$$

Now, let $\{\hat{\mathbf{v}}_i\}$ be the dual basis of $V_h(\hat{E})$ to the degrees of freedom, and let

$$\hat{a}_i = g|_{f_i}|J|_{f_i}||\hat{f}_i|, \qquad i = 1, \ldots, 5. \tag{17}$$

and

$$\hat{\mathbf{v}} = \sum_{i=1}^{5} \hat{a}_i \hat{\mathbf{v}}_i.$$

Then, from (10) applied to the element $\hat{E}$ we know that

$$c_{\hat{E}}^2 \|\hat{\mathbf{v}}\|_{L^2(\hat{E})}^2 \leq \frac{1}{h_{\hat{E}}} \sum_{i=1}^{5} \hat{a}_i^2 \leq C_{\hat{E}}^2 \|\hat{\mathbf{v}}\|_{L^2(\hat{E})}^2 \tag{18}$$

and using (13) for $\hat{E}$ instead of E we have

$$\|\hat{\mathbf{v}}\|_{L^2(\hat{E})} = \sup \left\{ \int_{\partial \hat{E}} \hat{g}|J|\hat{q} : \quad \hat{q} \in H^1(\hat{E}), \|\nabla \hat{q}\|_{L^2(\hat{E})} = 1, \int_{\hat{E}} \hat{q} = 0 \right\}.$$

It follows from (18) that

$$\left(\frac{1}{h_{\hat{E}}} \sum_{i=1}^{5} \hat{a}_i^2 \right)^{\frac{1}{2}} \sim \sup \left\{ \int_{\partial \hat{E}} \hat{g}|J|\hat{q} : \quad \hat{q} \in H^1(\hat{E}), \|\nabla \hat{q}\|_{L^2(\hat{E})} = 1, \int_{\hat{E}} \hat{q} = 0 \right\},$$

where the constants in this equivalence depend on the aspect ratio of E, and so, since $h_{\hat{E}} \sim 1$, this equation together with (15) and (16) gives

$$\frac{1}{h_E} \sum_{i=1}^{5} \hat{a}_i^2 \sim \|\mathbf{v}\|_{L^2(E)}^2.$$

But, from (12) and (17) and the definition of $|J|$

$$\hat{a}_i = a_i \frac{|J|_{f_i}|}{|f_i|} |\hat{f}_i| = a_i, \qquad i = 1, \ldots, 5,$$

and then we obtain

$$\frac{1}{h_E} \sum_{i=1}^{5} a_i^2 \sim \|\mathbf{v}\|_{L^2(E)}^2$$

as we wanted, since the constants in this equivalence depend only on the aspect ratio of E. $\square$

As a consequence of this result and the definition of the bilinear form $\mathcal{S}^E$ we obtain the next Corollary.

Corollary 3.3. *If E is a pyramid, we have*

$$c_E a^E(\mathbf{v}, \mathbf{v}) \leq h_E^{-1} \mathcal{S}^E(\mathbf{v}, \mathbf{v}) \leq C_E a^E(\mathbf{v}, \mathbf{v}) \qquad \forall \mathbf{v} \in V_h(E), \tag{19}$$

where the constants c_E and C_E depend only on the shape regularity of E.

Lemma 3.4. *For all element E we have*

$$a_h^E(\mathbf{u}, \mathbf{v}) = a^E(\mathbf{u}, \mathbf{v}), \qquad \forall E \in \mathcal{T}_h, \quad \forall \mathbf{u} \in W(E), \quad \forall \mathbf{v} \in V_h(E), \tag{20}$$

$$c_E a^E(\mathbf{v}, \mathbf{v}) \leq a_h^E(\mathbf{v}, \mathbf{v}) \leq C_E a^E(\mathbf{v}, \mathbf{v}) \qquad \forall \mathbf{v} \in V_h(E) \tag{21}$$

with $c_E = C_E = 1$ when E is a tetrahedron or a right prism, and c_E and C_E depend on the shape regularity of E when E is a pyramid.

Proof. The statement follows easily from Corollary 3.3 and Remark 3.1. $\square$

Define
$$\mathcal{K}_h = \{\mathbf{v}_h \in V_h : b(\mathbf{v}_h, q) = 0 \ \forall q \in Q_h\}.$$
We note that
$$\mathcal{K}_h = \{\mathbf{v}_h \in V_h : \operatorname{div} \mathbf{v}_h = 0\} \subset \{\mathbf{v} \in V : \operatorname{div} \mathbf{v} = 0\} := \mathcal{K}.$$
The next result is immediately obtained.

Proposition 3.5. *The discrete bilinear form a_h is coercive on $\mathcal{K}_h$ and the coercivity constant depends only on the shape regularity of the pyramids of the mesh.*

Proposition 3.6. *The local discrete bilinear form a_h^E is continuous in $V_h(E)$, that is,*
$$\left|a_h^E(\mathbf{u}, \mathbf{v})\right| \leq C\|\mathbf{u}\|_{L^2(E)}\|\mathbf{v}\|_{L^2(E)}, \qquad \forall \mathbf{u}, \mathbf{v} \in V_h(E),$$
where C equals 1 when E is a right prism or tetrahedron, and depends only on the aspect ratio of E in the case of pyramids.

Proof. When E is a prism or tetrahedron, $a_h^E(\mathbf{u}, \mathbf{v}) = a^E(\mathbf{u}, \mathbf{v})$ for $\mathbf{u}$ and $\mathbf{v}$ in $V_h(E)$, and the result is clear from the definition of a^E. When E is a pyramid, we observe that a_h^E is symmetric and coercive in $L^2(E)$ because of (21), and then it defines an inner product. Hence, from Cauchy-Schwarz inequality and (21) again we have
$$\left|a_h^E(\mathbf{u}, \mathbf{v})\right| \leq a_h^E(\mathbf{u}, \mathbf{u})^{\frac{1}{2}} a_h^E(\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \leq C_E a^E(\mathbf{u}, \mathbf{u})^{\frac{1}{2}} a^E(\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \leq C_E\|\mathbf{u}\|_{L^2(E)}\|\mathbf{v}\|_{L^2(E)}.$$
The constant C_E depends on the aspect ratio of E. This concludes the proof. □

Finally, we define the discrete problem: To find $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that
$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) &= 0 \qquad \forall \mathbf{v} \in V_h \\ b(\mathbf{u}_h, q) &= (f, q) \qquad \forall q \in Q_h. \end{aligned} \tag{22}$$

4. The space $W(E)$

We know that when E is a tetrahedron or a prism $W(E) = V_h(E)$. The purpose of this Section is to characterize $W(E)$ when E is a pyramid. The Section finishes with some computational insights.

We start with the next Lemma that can be easily proved.

Lemma 4.1. *Let $\hat{P}$ be the reference pyramid with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,0,1)$. We denote by $\hat{f}_1$ the face with vertices at $(0,0,0)$, $(0,1,0)$, $(0,0,1)$, by $\hat{f}_2$ the one with vertices at $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, by $\hat{f}_3$ the one with vertices at $(1,0,0)$, $(1,1,0)$, $(0,0,1)$, by $\hat{f}_4$ the one with vertices at $(1,1,0)$, $(0,1,0)$, $(0,0,1)$ and by $\hat{f}_5$ the square basis.*

Then if $\mathbf{v} \in \mathcal{P}_1(\hat{P})^3$ verifies $\mathbf{v} \cdot \mathbf{n} = 0$ on $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$ and $\hat{f}_5$, then $\mathbf{v}(\mathbf{x}) = (0, cx_2, 0)$ with c constant.

Lemma 4.2. *Let P be a pyramid. Then $\dim W(P) \leq 4$.*

Proof. We have $W(P) \subseteq V(P)$ and $\dim V(P) = 5$. In order to prove that $W(P) \neq V(P)$ we will show that there exists no field $\mathbf{v} = \nabla q_2$ with $q_2 \in \mathcal{P}_2(P)$ with vanishing normal component on four faces of P and being constant different from 0 on the other face.

Let $\hat{P}$ be the reference pyramid of Lemma 4.1 and use the same notation for the faces. Let $F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}} + \mathbf{b}$ be an affine map from $\hat{P}$ onto P and we denote $f_i = F(\hat{f}_i)$. Suppose that $\mathbf{v} = \nabla q_2 \in \mathcal{P}_2(P)$ is such that $\mathbf{v} \cdot \mathbf{n} = 0$ on f_1, f_2, f_3 and f_5, while $\mathbf{v} \cdot \mathbf{n} = 1$ on f_4. Now we consider $\hat{\mathbf{v}}$ obtained via the Piola trasform from $\mathbf{v}$, that is
$$\mathbf{v}(\mathbf{x}) = \frac{1}{|B|} B\hat{\mathbf{v}}(\hat{\mathbf{x}}), \qquad \mathbf{x} = F(\hat{\mathbf{x}}), \tag{23}$$

which is in $\mathcal{P}_1(\hat{P})^3$. Using properties of the Piola transform [9, pages 12–14] we have for $i = 1, 2, 3, 5$,

$$\int_{\hat{f}_i} \hat{\mathbf{v}} \cdot \mathbf{n}\hat{\phi} = \int_{f_i} \mathbf{v} \cdot \mathbf{n}\phi = 0 \qquad \forall \phi \in \mathcal{P}_1(f_i).$$

with $\hat{\phi} = \phi \circ F$. Since $\hat{\mathbf{v}}|_{\hat{f}_i} \cdot \mathbf{n} \in \mathcal{P}_1(\hat{f}_i)$, this implies $\hat{\mathbf{v}}|_{\hat{f}_i} \cdot \mathbf{n} = 0$ for $i = 1, 2, 3, 5$. From the previous Lemma we obtain that

$$\hat{\mathbf{v}}(\hat{\mathbf{x}}) = (0, c\hat{x}_2, 0).$$

Then

$$\mathbf{v}(\mathbf{x}) = \frac{1}{|B|} B(0, c\hat{x}_2, 0)^t = \frac{c}{|B|}\mathbf{b}_2\hat{x}_2$$

where $\mathbf{b}_2$ is the second column of B. Then, on f_4 we have

$$\mathbf{v}(\mathbf{x}) \cdot \mathbf{n} = \frac{c}{|B|}\hat{x}_2\mathbf{b}_2 \cdot \mathbf{n}$$

and we note that $\hat{x}_2$ is not constant on f_4, it varies from 0 to 1, and $\mathbf{b}_2 \cdot \mathbf{n} \neq 0$, since $\mathbf{b}_2$ is a transversal vector to the face f_4. Then, $\mathbf{v}(\mathbf{x}) \cdot \mathbf{n}$ is not constant on f_4, which contradicts our definition of $\mathbf{v}$. $\square$

Proposition 4.3. *Let P be a pyramid. Then $W(P) = \mathcal{P}_0^3(P) + \mathbf{x}\mathcal{P}_0(P)$.*

Proof. We have $P_0^3(P) + \mathbf{x}\mathcal{P}_0(P) \subseteq W(P)$. Since $\dim\big(\mathcal{P}_0^3(P) + \mathbf{x}\mathcal{P}_0(P)\big) = 4$ and that from Lemma 4.2, $\dim W(P) \leq 4$, we conclude the assertion. $\square$

Given a field $\mathbf{v} \in V_h(E)$, we can construct $\Pi_w^E\mathbf{v}$ as follows. We choose a basis $\{\mathbf{w}_i\}$ of $W(E)$, for example,

$$\{(1,0,0), (0,1,0), (0,0,1), (x,y,z)\} =: \{\mathbf{w}_i : 1 \leq i \leq 4\}$$

with $\mathbf{w}_i = \nabla q_i$, $i = 1, 2, 3, 4$. Then $a^E(\mathbf{v}, \mathbf{w}_i)$ is calculable from $\mathbf{v}$'s degrees of freedom

$$a^E(\mathbf{v}, \mathbf{w}_i) = \int_E \mathbf{v} \cdot \nabla q_i = -\int_E \operatorname{div}\mathbf{v}\, q_i + \int_{\partial E} \mathbf{v} \cdot \mathbf{n}\, q_i.$$

Then if $\Pi_w^E\mathbf{v} = \sum_{j=1}^4 \alpha_j \mathbf{w}_j$ we can compute the coefficients α_j by solving the linear system

$$\sum_{j=1}^4 \alpha_j a^E(\mathbf{w}_j, \mathbf{w}_i) = a^E(\mathbf{v}, \mathbf{w}_i), \qquad i = 1, 2, 3, 4.$$

In order to compute the stabilization part of the discrete bilinear form we need to write $\Pi_w^E\mathbf{v}$ as a linear combination of the basis $\{\mathbf{v}_i\}$ of $V_h(E)$ associated with the degrees of freedom, this is (always in the pyramidal case)

$$\int_{f_i} \mathbf{v}_j \cdot \mathbf{n} = \delta_{ij}, \qquad 1 \leq i, j \leq 5.$$

In this case, we have $\Pi_w^E\mathbf{v} = \sum_{i=1}^5 \beta_i\mathbf{v}_i$, with

$$\beta_i = \sum_{j=1}^4 \alpha_j \int_{f_i} \mathbf{w}_j \cdot \mathbf{n}.$$

5. Local interpolation error estimates

In this section we obtain estimates for the local interpolation errors $\|\mathbf{u}-\mathbf{u}_I\|_{L^2(E)}$. In the case of prismatic elements, these estimates are anisotropic and valid on arbitrarily narrow prisms. For the other shapes, the estimates depend on the aspect ratio of the elements (in the case of tetrahedra, this is just for simplicity, see [1]). For pyramids we cannot follow standard re–scaling arguments as in [1] since the local discrete spaces are not preserved by the Piola transformation due to the **curl**–vanishing condition.

We start this Section by analyzing interpolation error estimates in triangular right prisms. As we mentioned in the Introduction, anisotropic interpolation error estimates for the Raviart–Thomas operator of lowest order on right prisms were analyzed in [13]. There, the authors deduce interpolation error estimates for functions in weighted Sobolev spaces which are adequate to deal with elliptic problems in prismatic domains. Since we need to use that kind of estimates in a slightly more general situation, involving weights coming also from vertex singularities, and therefore we cannot use directly the results of [13], and also for the sake of completeness, we deduce here the anisotropic error estimates in the form that we will use in Section 7. Furthermore, this approach can be easily extended to higher order interpolations, see [16] for further details.

We denote by $\hat{P}$ the reference prism with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$ and $(0,1,1)$. Furthermore we denote by $\hat{f}_1,\hat{f}_2,\hat{f}_3,\hat{f}_4$ and $\hat{f}_5$ the faces of $\hat{P}$ with outer normal equal, respectively, to $(-1,0,0)$, $(0,-1,0)$, $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, $(0,0,-1)$ and $(0,0,1)$. Given an element E, for a field $\mathbf{v}\in W^{1,1}(E)^3$ we recall that $\mathbf{v}_I$ denotes the $V_h(E)$-interpolation defined in Section 2. We denote by $\mathbf{v}_{I,i}$, $i=1,2,3$ the components of $\mathbf{v}_I$.

Lemma 5.1. *Let $\hat{\mathbf{u}}\in W^{1,1}(\hat{P})^3$ be such that $\hat{u}_3=0$. Then $\hat{\mathbf{u}}_{I,3}=0$.*

Proof. First we note that $\hat{\mathbf{u}}_{I,3}$ is a linear polynomial of the variable x_3, we denote it by $p_1(x_3)$. Furthermore, $\hat{\mathbf{u}}_I|_{\hat{f}_4}\cdot\mathbf{n}=-p_1(0)$ and $\hat{\mathbf{u}}_I|_{\hat{f}_5}\cdot\mathbf{n}=p_1(1)$, and since $\hat{u}_3=0$ we have

$$\int_{\hat{f}_4}\hat{\mathbf{u}}\cdot\mathbf{n}=0,\qquad \int_{\hat{f}_5}\hat{\mathbf{u}}\cdot\mathbf{n}=0.$$

Then $p_1(0)=p_1(1)=0$ and so $p_1(x_3)\equiv 0$. □

Lemma 5.2. *If $\hat{\mathbf{u}}(\mathbf{x})=(u_1(x_2,x_3),0,0)\in W^{1,1}(\hat{P})^3$ then $\hat{\mathbf{u}}_I(\mathbf{x})=(a,0,0)$ with a constant. If $\hat{\mathbf{u}}(\mathbf{x})=(0,u_2(x_1,x_3),0)\in W^{1,1}(\hat{P})^3$ then $\hat{\mathbf{u}}_I(\mathbf{x})=(0,b,0)$ with b constant.*

Proof. Suppose that $\hat{\mathbf{u}}(\mathbf{x})=(u_1(x_2,x_3),0,0)\in W^{1,1}(\hat{P})$. Then we know from Lemma 5.1 that $\hat{\mathbf{u}}_I(\mathbf{x})=(p(x_1),q(x_2),0)$ with $p(x_1)=a+\gamma x_1$ and $q(x_2)=b+\gamma x_2$. From

$$-b|\hat{f}_2|=-\int_{\hat{f}_2}q=\int_{\hat{f}_2}\hat{\mathbf{u}}_I\cdot\mathbf{n}=\int_{\hat{f}_2}\hat{\mathbf{u}}\cdot\mathbf{n}=0$$

we obtain that $b=0$. Also we have

$$2\gamma=\operatorname{div}\hat{\mathbf{u}}_I=P_0(\operatorname{div}\hat{\mathbf{u}})=0.$$

Then $q(x_2)=b+\gamma x_2=0$ and $p(x_1)=a+\gamma x_1=a$. □

Lemma 5.3. *Let $\hat{\mathbf{u}}=(0,0,\hat{u}_3)\in W^{1,1}(\hat{P})^3$. Then $\hat{\mathbf{u}}_{I,1}=\hat{\mathbf{u}}_{I,2}=0$.*

Proof. We know that $\hat{\mathbf{u}}_I=(p(x_1),q(x_2),r(x_3))$ with $p(x_1)=a+\gamma x_1$ and $q(x_2)=b+\gamma x_2$, with real numbers a, b and γ. But, $\hat{u}_1=0$ and $\hat{u}_2=0$ imply

$$0=\int_{\hat{f}_1}\hat{\mathbf{u}}\cdot\mathbf{n}=\int_{\hat{f}_1}p=a,\qquad\text{and}\qquad 0=\int_{\hat{f}_2}\hat{\mathbf{u}}\cdot\mathbf{n}=\int_{\hat{f}_2}p=b,$$

and then

$$0 = \int_{\hat{f}_3} \hat{\mathbf{u}} \cdot \mathbf{n} = \int_{\hat{f}_3} \hat{\mathbf{u}}_I \cdot \mathbf{n} = \gamma,$$

which concludes the proof. □

Lemma 5.4. *There exists a constant C such that for all $\mathbf{u} \in W^{1,1}(\hat{P})$ we have*

$$\|\mathbf{u}_{I,1}\|_{L^1(\hat{P})} \leq C\left(\|u_1\|_{W^{1,1}(\hat{P})} + \|\mathrm{div}\,(u_1,u_2,0)\|_{L^1(\hat{P})}\right) \tag{24}$$

$$\|\mathbf{u}_{I,2}\|_{L^1(\hat{P})} \leq C\left(\|u_2\|_{W^{1,1}(\hat{P})} + \|\mathrm{div}\,(u_1,u_2,0)\|_{L^1(\hat{P})}\right) \tag{25}$$

$$\|\mathbf{u}_{I,3}\|_{L^1(\hat{P})} \leq C\|u_3\|_{W^{1,1}(\hat{P})}. \tag{26}$$

Proof. From Lemma 5.3 we have $\mathbf{u}_{I,1} = (u_1,u_2,0)_{I,1}$. From this and Lemma 5.2 we have that for $\mathbf{v}(\mathbf{x}) := (u_1, u_2 - u_2(x_1,0,x_3), 0)$ it holds

$$\mathbf{u}_{I,1} = \mathbf{v}_{I,1},$$

so it suffices to estimate $\|\mathbf{v}_{I,1}\|_{L^1(\hat{P})}$. But $\mathbf{v}_{I,1}$ is defined by the degrees of freedom

$$dof_i = \int_{\hat{f}_i} \mathbf{v} \cdot \mathbf{n}, \qquad i = 1,\ldots,5.$$

Taking each one of them we have

$$|dof_1| = \left|-\int_{\hat{f}_1} v_1\right| = \left|-\int_{\hat{f}_1} u_1\right| \leq C\|u_1\|_{W^{1,1}(\hat{P})} \tag{27}$$

$$|dof_2| = \left|-\int_{\hat{f}_2} v_2\right| = 0. \tag{28}$$

For dof_3, using the Divergence Theorem on $\hat{P}$ and taking into account that v_2 vanishes on $\hat{f}_2$, we have

$$\begin{aligned} dof_3 &= \frac{1}{\sqrt{2}}\int_{\hat{f}_3} (v_1 + v_2) \\ &= \int_{\hat{P}} \mathrm{div}\,\mathbf{v} + \int_{\hat{f}_1} v_1 \\ &= \int_{\hat{P}} \mathrm{div}\,(u_1,u_2,0) + \int_{\hat{f}_1} u_1, \end{aligned}$$

and then we obtain

$$|dof_3| \leq C\|u_1\|_{W^{1,1}(\hat{P})} + \|\mathrm{div}\,(u_1,u_2,0)\|_{L^1(\hat{P})}. \tag{29}$$

Finally we observe that we have

$$dof_4 = dof_5 = 0. \tag{30}$$

Inequalities (27)–(30) imply estimate (24).

Estimate (25) follows analogously. For (26) we note that if $\mathbf{w} = (0,0,u_3)$ then

$$\mathbf{u}_{I,3} = \mathbf{w}_{I,3},$$

while all the degrees of freedom defining $\mathbf{w}_I$ are bounded by a constant times $\|u_3\|_{W^{1,1}(\hat{P})}$, which gives (26). □

Given a general right prism P, we call $\mathtt{v}_i$, $i = 0, \ldots, 5$ its vertices, $\mathtt{v}_0\mathtt{v}_1\mathtt{v}_2$ and $\mathtt{v}_3\mathtt{v}_4\mathtt{v}_5$ being its triangular bases, so that $\mathtt{v}_0\mathtt{v}_3, \mathtt{v}_1\mathtt{v}_4$ and $\mathtt{v}_2\mathtt{v}_5$ are parallel segments which are perpendicular to the triangular faces. We consider a Cartesian system of coordinates with the x_3-axis parallel to $\mathtt{v}_0\mathtt{v}_3$. Furthermore we define $h_{P,i} = |\mathtt{v}_0\mathtt{v}_i|$, $\xi_{P,i} = \frac{\mathtt{v}_0\mathtt{v}_i}{h_{P,i}}$, $i = 1, 2, 3$, and denote by α_P the maximum angle of the triangle $\mathtt{v}_0\mathtt{v}_1\mathtt{v}_2$.

Let $\hat{P}$ be the reference prism (in the Cartesian system of coordinates considered before). Then there exists a linear transformation $F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}} + \mathbf{b}$ which sends $\hat{P}$ onto P, with the matrix B having the form

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}. \tag{31}$$

To each field $\hat{\mathbf{v}}$ on $\hat{P}$ we associate the corresponding field $\mathbf{v}$ on P defined by (23) using the Piola Transform. Then it can be seen that

$$V_h(P) = \left\{ \mathbf{v} : \hat{\mathbf{v}} \in V_h(\hat{P}) \right\}.$$

We recall the properties

$$\operatorname{div}(v_1(\mathbf{x}), v_2(\mathbf{x}), 0) = \frac{1}{|B|} \operatorname{div}(\hat{v}_1(\hat{\mathbf{x}}), \hat{v}_2(\hat{\mathbf{x}}), 0).$$

and

$$\mathbf{v}_I(\mathbf{x}) = \frac{1}{|B|} B \hat{\mathbf{v}}_I(\hat{\mathbf{x}}).$$

The next Lemma is in the same spirit of [13, Lemma 3.1], being, in our case, valid for functions in $W^{1,1}(P)$.

Lemma 5.5. *Let P be a right prism. There exists a constant C depending only on α_P such that for all $\mathbf{u}$ in $W^{1,1}(P)$ we have*

$$\|\mathbf{u}_I\|_{L^1(P)} \le C \left(\|\mathbf{u}\|_{L^1(P)} + \sum_{i=1}^{3} h_{i,P} \|\partial_{\xi_{P,i}} \mathbf{u}\|_{L^1(P)} + \max\{h_{P,1}, h_{P,2}\} \|div\,(u_1, u_2, 0)\|_{L^1(P)} \right). \tag{32}$$

Proof. Using the notation introduced above for the vertices of P, suppose that $\mathtt{v}_0$ is the vertex with the maximum angle of the triangle $\mathtt{v}_0\mathtt{v}_1\mathtt{v}_2$. Let $\tilde{P}$ be a prism with vertices at $(0,0,0)$, $(h_{P,1}, 0, 0)$, $(0, h_{P,2}, 0)$, $(0, 0, h_{P,3})$, $(h_{P,1}, 0, h_{P,3})$ and $(0, h_{P,2}, h_{P,3})$. Then by standard rescaling arguments using the Piola Transform we can prove from Lemma (5.4) that there exists a constant C such that for all $\tilde{\mathbf{u}} \in W^{1,1}(\tilde{P})$ we have

$$\|\tilde{\mathbf{u}}_I\|_{L^1(\tilde{P})} \le C \left(\|\tilde{\mathbf{u}}\|_{L^1(\tilde{P})} + \sum_{i=1}^{3} h_{P,i} \|\partial_{x_i} \tilde{\mathbf{u}}\|_{L^1(\tilde{P})} + \max\{h_{P,1}, h_{P,2}\} \|\operatorname{div}(\tilde{u}_1, \tilde{u}_2, 0)\|_{L^1(\tilde{P})} \right).$$

Let B be the matrix with columns $\xi_{P,1}$, $\xi_{P,2}$ and $\xi_{P,3}$ (note B has the form (31) and $\xi_{P,3} = (0,0,1)$). Then the map $F(\tilde{\mathbf{x}}) = B\tilde{\mathbf{x}} + \mathtt{v}_0$ sends $\tilde{P}$ onto P. Then, again by a change of variables, we obtain from the previous estimate, that for all $\mathbf{u} \in W^{1,1}(P)$ it holds

$$\begin{aligned} \|\mathbf{u}_I\|_{L^1(P)} \le\; & C\|B\|\|B^{-1}\| \Bigg(\|\mathbf{u}\|_{L^1(P)} + \sum_{i=1}^{3} h_{P,i} \|\partial_{\xi_{P,i}} \mathbf{u}\|_{L^1(P)} \\ & + \max\{h_{P,1}, h_{P,2}\} \frac{1}{\|B^{-1}\|} \|\operatorname{div}(u_1, u_2, 0)\|_{L^1(P)} \Bigg). \end{aligned}$$

Then the proof concludes by noting that $\|B\| \le C$ and $\|B^{-1}\| \sim \sin \alpha_P$. $\square$

Remark 5.6. *Stability estimates in L^p-norm, $p > 1$, can be proved analogously. In particular, from (32), using an inverse inequality on the left hand side, and Cauchy-Schwarz inequality on the right hand side, we obtain under assumptions of Lemma 5.5*

$$\|\mathbf{u}_I\|_{L^2(P)} \le C\left(\|\mathbf{u}\|_{L^2(P)} + \sum_{i=1}^{3} h_{i,P}\|\partial_{\xi_{P,i}}\mathbf{u}\|_{L^2(P)} + \max\{h_{P,1}, h_{P,2}\}\|\mathrm{div}\,(u_1, u_2, 0)\|_{L^2(P)}\right). \tag{33}$$

Under the additional assumption $h_{P,3} \ge \max\{h_{P,1}, h_{P,2}\}$ we have

$$\|\mathbf{u}_I\|_{L^2(P)} \le \left(\|\mathbf{u}\|_{L^2(P)} + \sum_{i=1}^{3} h_{P,i}\|\partial_{\xi_i}\mathbf{u}\|_{L^2(P)} + h_P\|\mathrm{div}\,\mathbf{u}\|_{L^2(P)}\right). \tag{34}$$

The next interpolation error estimate can be viewed as an extension of [13, Theorem 3.2]: it is valid for functions in $H^1(P)$ but paying the price of requiring that the height has to be longer than the diameter of the basis of P .

Proposition 5.7. *Let P be a right prism, and consider a local system of coordinates $x_1x_2x_3$ such that the triangular bases of P are parallel to the x_1x_2-coordinate plane. Denote by $\xi_{P,1}$ and $\xi_{P,2}$ the versors parallel to the edges of the triangular bases of P adjacent to its maximum angle α_P, $\xi_{P,3} = (0, 0, 1)$ and $h_{P,i}$ are the lengths of the edges of P parallel to $\xi_{P,i}$. We assume that $h_{P,3} > ch_{P,1}$ and $h_{P,3} > ch_{P,2}$. Then, there exists a constant C depending only on c and α_P, such that for all $\mathbf{u} \in H^1(P)$ we have*

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(P)} \le C\left(\sum_{i=1}^{3} h_{P,i}\|\partial_{\xi_{P,i}}\mathbf{u}\|_{L^2(P)} + h_T\|\mathrm{div}\,\mathbf{u}\|_{L^2(P)}\right). \tag{35}$$

Proof. Let $Q\mathbf{u}$ be the average of $\mathbf{u}$ on P. Then we can write

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(P)} \le \|\mathbf{u} - Q\mathbf{u}\|_{L^2(P)} + \|Q\mathbf{u} - \mathbf{u}_I\|_{L^2(P)} = \|\mathbf{u} - Q\mathbf{u}\|_{L^2(P)} + \|(\mathbf{u} - Q\mathbf{u})_I\|_{L^2(P)}. \tag{36}$$

We recall the following estimate from [1]

$$\|\mathbf{u} - Q\mathbf{u}\|_{L^2(P)} \le C\sum_{i=1}^{3} h_{P,i}\|\partial_{\xi_{P,i}}\mathbf{u}\|_{L^2(P)}.$$

This estimate together with (34) gives (35). □

A result similar to the next one, can be found in [7]. Since, the proof in our case is simpler and for the sake of completeness we include it below.

Proposition 5.8. *If E is an isotropic tetrahedron or pyramid, then*

$$\|\mathbf{u}_I\|_{L^p(E)} \le C\left(\|\mathbf{u}\|_{L^p(E)} + h_E|\mathbf{u}|_{W^{1,p}(E)}\right), \qquad \forall \mathbf{u} \in W^{1,p}(E), \tag{37}$$

with the constant C depending on the aspect ratio of E, and $1 \le p$ if E is a tetrahedron and $1 \le p \le 2$ if E is a pyramid.

Proof. When E is a tetrahedron, this result is contained in [1]. So we assume that E is a pyramid. We note that

$$\mathbf{u}_I = \sum_{i=1}^{5}\left(\int_{f_i} \mathbf{u}\cdot\mathbf{n}\right)\mathbf{v}_i \tag{38}$$

where $\{\mathbf{v}_i\}_{i=1}^5$ is the basis of $V_h(E)$ dual to the degrees of freedom (7). Denote by f_j, $j = 1, \dots, 5$ the faces of E. First of all we need to estimate the L^2-norm of the basis functions $\mathbf{v}_i$. Fixed $1 \le i \le 5$, it follows from the proof of Lemma 2.1 that $\mathbf{v}_i = \nabla\psi$ where ψ is the solution of

$$\begin{aligned} \Delta\psi &= d \qquad \text{in } \Omega \\ \frac{\partial\psi}{\partial\mathbf{n}} &= g \qquad \text{on } \partial E \\ \int_E \psi &= 0 \end{aligned}$$

with

$$g|_{f_j} = \begin{cases} \frac{1}{|f_i|} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \qquad d = \frac{1}{|E|}.$$

Multiplying the first equation defining ψ by ψ, integrating by parts and using that $\int_E \psi = 0$, we obtain

$$\|\nabla\psi\|^2_{L^2(E)} = \int_{\partial E} g\psi \leq \|g\|_{L^2(\partial E)}\|\psi\|_{L^2(\partial E)}.$$

By a trace inequality we have, for a constant C depending on the aspect ratio of E

$$\|\nabla\psi\|^2_{L^2(E)} \leq C h_E^{\frac{1}{2}} \|g\|_{L^2(\partial E)}\|\nabla\psi\|_{L^2(E)}. \tag{39}$$

Taking into account the definition of g we have

$$\|g\|_{L^2(\partial E)} \leq C h_E^{-1}$$

and so from (39) we obtain

$$\|\mathbf{v}_i\|_{L^2(E)} = \|\nabla\psi\|_{L^2(E)} \leq C h_E^{-\frac{1}{2}}. \tag{40}$$

Now, for $1 \leq p \leq 2$ using Hölder's inequality and the expression (38) we have

$$\begin{aligned} \|\mathbf{u}_I\|_{L^p(E)} &\leq |E|^{\frac{1}{p}-\frac{1}{2}} \|\mathbf{u}_I\|_{L^2(T)} \\ &\leq |E|^{\frac{1}{p}-\frac{1}{2}} \sum_{i=1}^{5} \left| \int_{f_i} \mathbf{u}\cdot\mathbf{n} \right| \|\mathbf{v}_i\|_{L^2(E)}. \end{aligned}$$

By using (40), Hölder's inequality, trace inequalities and taking into account the shape-regularity of E we obtain

$$\begin{aligned} \|\mathbf{u}_I\|_{L^p(E)} &\leq C|E|^{\frac{1}{p}-\frac{1}{2}} h_E^{-\frac{1}{p}} \left(\|\mathbf{u}\|_{L^p(\Omega)} + h_E\|\nabla\mathbf{u}\|_{L^p(E)}\right) |\partial E|^{1-\frac{1}{p}} \|\mathbf{v}_i\|_{L^2(E)} \\ &\leq C h_E^{3\left(\frac{1}{p}-\frac{1}{2}\right)} h_E^{-\frac{1}{p}} h_E^{2\left(1-\frac{1}{p}\right)} h_E^{-\frac{1}{2}} \left(\|\mathbf{u}\|_{L^p(\Omega)} + h_E\|\nabla\mathbf{u}\|_{L^p(E)}\right) \\ &= C\left(\|\mathbf{u}\|_{L^p(\Omega)} + h_E\|\nabla\mathbf{u}\|_{L^p(E)}\right) \end{aligned}$$

where C depends on the shape regularity of E. □

Proposition 5.9. *Let E be a tetrahedron or a pyramid with aspect ratio σ. Then there exists a constant C depending only on σ such that*

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(E)} \leq C h_E |\mathbf{u}|_{H^1(E)} \qquad \forall \mathbf{u} \in H^1(E).$$

Proof. Let $Q\mathbf{u}$ be the $L^2(E)$-projection of u onto the constant fields. Then we have

$$\mathbf{u} - \mathbf{u}_I = (\mathbf{u} - Q\mathbf{u}) + (Q\mathbf{u} - \mathbf{u})_I$$

and using Proposition 5.8 and a clasical estimate for the $L^2(E)$-projection error we have

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_I\|_{L^2(E)} &\leq \|\mathbf{u}-Q\mathbf{u}\|_{L^2(E)} + \|(\mathbf{u}-Q\mathbf{u})_I\|_{L^2(E)} \\
&\leq \|\mathbf{u}-Q\mathbf{u}\|_{L^2(E)} + C\left(\|\mathbf{u}-Q\mathbf{u}\|_{L^2(E)} + h_E\|\nabla(\mathbf{u}-Q\mathbf{u})\|_{L^2(E)}\right) \\
&= C\left(\|\mathbf{u}-Q\mathbf{u}\|_{L^2(E)} + h_E\|\nabla\mathbf{u}\|_{L^2(E)}\right) \\
&\leq Ch_E\|\nabla\mathbf{u}\|_{L^2(E)}
\end{aligned}$$

as we wanted to prove. □

Proposition 5.10. *Let E be a pyramid with aspect ratio σ and $\mathbf{u} \in H^1(E)$. There exists a field $\mathbf{u}_\pi \in W(E)$ such that*

$$\|\mathbf{u}-\mathbf{u}_\pi\|_{L^2(E)} \leq Ch_E|\mathbf{u}|_{H^1(E)}$$

with C depending only on σ.

Proof. We can define $\mathbf{u}_\pi$ on E as the $L^2(E)$-projection of $\mathbf{u}$ onto the space of constant fields $P_0(E)^3 \subset W(E)$. The error estimate follows from Bramble-Hilbert Lemma. □

Lemma 5.11. *There exists a constant $\beta^* > 0$ depending only on Ω and the maximum aspect ratio of the pyramids of $\mathcal{T}_h$ such that for all $q^* \in Q_h$ there exists $\mathbf{w}_h^* \in V_h$ such that*

$$\mathit{div}\,\mathbf{w}_h^* = q^*, \qquad \beta^*\|\mathbf{w}_h^*\|_{L^2(\Omega)} \leq \|q^*\|_{L^2(\Omega)}$$

Proof. Since $q^* \in L^2(\Omega)$, we know [9, page 6] that there exists $\mathbf{w}^* \in [H^1(\Omega)]^3$ such that

$$\operatorname{div}\mathbf{w}^* = q, \quad \text{in } \Omega, \qquad \beta\|\mathbf{w}^*\|_{H^1(\Omega)} \leq \|q\|_{L^2(\Omega)}$$

with β^* depending only on Ω. Take $\mathbf{w}_h^* = \mathbf{w}_I^*$, the V_h-interpolant of $\mathbf{w}^*$. Then we see that

$$\|\mathbf{w}_h^*\|_{L^2(\Omega)} \leq C(1+h)\|\mathbf{w}^*\|_{H^1(\Omega)} \leq \frac{C(1+h)}{\beta}\|q^*\|_{L^2(\Omega)},$$

with C depending on the maximum aspect ratio of the pyramids of $\mathcal{T}_h$, and where we used stability estimates for RT_0 in the case of prismatic and tetrahedral (possibly anisotropic) elements and the stability estimate (37) for VEM on pyramids. On the other hand, since $q^* \in Q_h$,

$$\operatorname{div}\mathbf{w}_h^* = P_0^{\mathcal{T}_h}\operatorname{div}\mathbf{w}^* = P_0^{\mathcal{T}_h}q^* = q^*,$$

where $P_0^{\mathcal{T}_h}$ is the L^2-projection onto Q_h. This concludes the proof. □

Now, we can prove the discrete inf-sup condition. Let $q^* \in Q_h$, and $\mathbf{w}_h^* \in V_h$ with $\operatorname{div}\mathbf{w}_h^* = q^*$ and $\beta^*\|\mathbf{w}_h^*\|_{L^2(\Omega)} \leq \|q^*\|_{L^2(\Omega)}$. Then

$$\|\mathbf{w}_h^*\|^2_{H(\operatorname{div},\Omega)} = \|\mathbf{w}_h^*\|^2_{L^2(\Omega)} + \|q^*\|^2_{L^2(\Omega)} \leq \left(\frac{1}{(\beta^*)^2}+1\right)\|q^*\|^2_{L^2(\Omega)}$$

So,

$$\sup_{0\neq\mathbf{v}\in V_h}\frac{(q^*,\operatorname{div}\mathbf{v})}{\|\mathbf{v}\|_{H(\operatorname{div},\Omega)}} \geq \frac{(q^*,\operatorname{div}\mathbf{w}_h^*)}{\|\mathbf{w}_h^*\|_{H(\operatorname{div},\Omega)}} \geq \frac{1}{\sqrt{\frac{1}{(\beta^*)^2}+1}}\|q^*\|_{L^2(\Omega)}.$$

Then we proved that there exists a constant C which depends only on Ω and the maximum aspect ratio of the pyramids of $\mathcal{T}_h$ such that

$$\sup_{0\neq\mathbf{v}\in V_h}\frac{(q,\operatorname{div}\mathbf{v})}{\|\mathbf{v}\|_{H(\operatorname{div},\Omega)}} \geq C\|q\|_{L^2(\Omega)} \qquad \forall q \in Q_h. \tag{41}$$

6. The approximation error estimate

The error analysis for the method proposed here follows as in [10]. In particular we have the next Theorem.

Theorem 6.1. *The discrete problem* (22) *has a unique solution* $(\mathbf{u}_h, p_h)$*. Moreover, for every approximation* $\mathbf{u}_\pi$ *of* $\mathbf{u}$ *that is piecewise in* $W(E)$*,* $E \in \mathcal{T}_h$*, we have*

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} &\leq C\left(\|\mathbf{u}-\mathbf{u}_I\|_{L^2(\Omega)} + \|\mathbf{u}-\mathbf{u}_\pi\|_{L^2(\Omega)}\right) \\ \|p_I-p_h\|_{L^2(\Omega)} &\leq C\left(\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} + \|\mathbf{u}-\mathbf{u}_\pi\|_{L^2(\Omega)}\right) \end{aligned}$$

where C *is a constant independent of* h*, and* $p_I = P_0^{\mathcal{T}_h} p$*.*

Proof. The well posedness of (22) follows from the discrete inf-sup condition (41) and Proposition 3.5. The rest of the proof follows as the proof of Theorem 5.1 in [10], using the definitions of the discrete spaces V_h and Q_h, the property (9), the inf-sup condition for b, the coercivity of a_h, the continuity of a_h (Proposition 3.6) and the definitions (6) and (22) of the continuous and the discrete problems. □

If Ω is convex and $f \in L^2(\Omega)$ we know that the solution p of problem (3) is in $H^2(\Omega)$, and so $\mathbf{u} \in H^1(\Omega)^3$. In order to estimate the approximation error in this case, we add to assumptions G1 and G2 stated in Section 2, a third geometrical condition concerning the shape of prismatic elements:

- G3 The triangular bases of all prismatic element in $\mathcal{T}_h$ have angles less than a constant $\alpha_r < \pi$, and heights of these elements are greater than a constant c_r times their respective diameters (see assumptions of Proposition 5.7).

In this case, using the interpolation error estimates we proved in Propositions 5.7 and 5.9, the result of Proposition 5.10, and a standard estimate for L^2-projection error, we obtain from Theorem 6.1 that

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} &\leq Ch|p|_{H^2(\Omega)} \\ \|p-p_h\|_{L^2(\Omega)} &\leq Ch|p|_{H^2(\Omega)} \end{aligned}$$

where the constant C depends only on σ_r of G2, and on α_r and c_r of G3. Arbitrarily narrow right prisms can be used in the mesh without affecting this estimate. This fact is further exploited in the next Section to deal with the case in which the domain Ω is not convex.

7. Error estimates for non–convex domains

In order to obtain optimal interpolation error estimates for the solution of (6) with $f \in L^2(\Omega)$ and Ω being a non–convex polyhedron we need further specific properties on the meshes in addition to G1 and G2, particularly, meshes will be graded towards the singularities of the domain yielding a family of graded meshes $\mathcal{T}_h$ with $h = \frac{1}{n}, n \in \mathbb{N}$. Along this Section we fix n. Recalling the definitions given at the end of the Introduction, we assume that Ω is split into macroelements satisfying:

- S1 Macroelements are tetrahedra or prisms denoted by Λ_ℓ, $\ell = 1, \ldots, M$ and are fixed when the process starts. We have $\overline{\Omega} = \cup_{\ell=1}^M \overline{\Lambda_\ell}$ and $\Lambda_{\ell_1} \cap \Lambda_{\ell_2} = \emptyset$ if $\ell_1 \neq \ell_2$.

We say that a macroelement contains a singular vertex or a singular edge when one of its vertices is a singular vertex of Ω and when it has an edge on a singular edge of Ω, respectively.

- S2 Each macroelement contains at most a singular vertex and at most a singular edge, and tetrahedral macroelements with a singular edge have a face orthogonal to that edge. When a tetrahedral macroelement contains both singular edge and vertex, then the singular vertex is opposite to the face perpendicular to the singular edge (and then it is an end point of that face).
- S3 Prismatic macroelement do not contain singular vertices.

In Section 8 we give guidelines to construct conforming meshes of Ω in order to obtain optimal approximation error estimates when singularities are present. In particular, each macroelement will be meshed in a way depending on whether its closure touches singular vertices or edges. In particular we have the next Theorem whose proof is postponed to Section 8. We denote $d(K,S)$ the distance between K and S.

Theorem 7.1. *Let Ω be a polyhedron with a splitting into macroelements $\Omega = \cup_{\ell=1}^{M} \Lambda_\ell$ satisfying conditions S1–S3. Let $h = \frac{1}{n}$ with $n \in \mathbb{N}$. Then, there exists a mesh $\mathcal{T}_h$ of Ω according to G1–G3, whose total number N of elements satisfies $N \leq Cn^3$, and such that the following conditions hold:*

(1) If an element K belongs to a macroelement with a singular edge e, then for some $0 < \mu < \lambda_e$ we have

$$h_{1,K}, h_{2,K} \lesssim \begin{cases} h^{\frac{1}{\mu}} & \text{if } d(K,e) = 0 \\ hd(K,e)^{1-\mu} & \text{if } 0 < d(K,e) < 1 \\ h & \text{if } 1 \leq d(K,e) \end{cases} \tag{42}$$

(2) If an element K belongs to a macroelement with a singular vertex v, then for some $0 < \nu < \lambda_v + \frac{1}{2}$ we have

$$h_{3,K} \lesssim \begin{cases} h^{\frac{1}{\nu}} & \text{if } d(K,v) = 0 \\ hd(K,v)^{1-\nu} & \text{if } 0 < d(K,v) < 1 \\ h & \text{if } 1 \leq d(K,v) \end{cases} \tag{43}$$

If K contains both singular vertex and edge, it is taken $\mu \leq \nu < 1$.

We say that the mesh of a macroelement satisfying condition (1) is graded toward the edge e, while the mesh of a macroelement satisfying (2) is graded toward the vertex v.

Remark 7.2. *We remark that if a mesh satisfies conditions (1) and (2) of Theorem 7.1 for $\mu = \mu_0$ and $\nu = \nu_0$, then it satisfies the same for $\mu > \mu_0$ and $\nu > \nu_0$. Thus, the mesh can be constructed for $\mu_0 = \nu_0 < \min\left\{\lambda_e, \lambda_v + \frac{1}{2}\right\}$ and it still verifies (42) and (43). The possibility to use $\mu_0 = \nu_0$ for the construction of the mesh, allows to validate assumption G2.*

Remark 7.3. *Suppose that a macroelement with a singular edge e and a singular vertex v is meshed in such a way that (42) and (43) are verified for $\mu = \mu_0$, $\nu = \nu_0$, with $\mu_0 = \nu_0 < \min\left\{\lambda_e, \lambda_v + \frac{1}{2}\right\}$. Then it can be easily checked that there exists μ_e, ν_v, δ_e and β_v such that $\mu_e \leq \nu_e$, $\mu_e \leq 1 - \delta_e$, $\nu_v \leq 1 - \beta_v$, $\delta_e > 1 - \lambda_e$ and $\beta_v > \frac{1}{2} - \lambda_v$ and (42) and (43) are satisfied for $\mu = \mu_e$ and $\nu = \nu_e$. Similar remarks can be done for cases of macroelements with only one kind of singularity. In summary, if necessary, we can assume that the meshes are refined more strongly than needed, and this is in order to verify condition G2. Of course, this does not affect the number of elements.*

Theorem 7.4. *Suppose that the family of meshes $\mathcal{T}_h$ satisfies conditions of Theorem 7.1 and is designed following Remark 7.3. Then we have the interpolation error estimate*

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}$$

Proof. For the part of the proof concerning edge singularities we could refer to [13]. However in order to simplify the exposition without introducing new notation and avoiding compatibility issues, and with the aim to make the article selfcontained, we expose here the full proof of this Theorem.

Taking into account the decomposition of $\mathbf{u}$ introduced in Theorem 1.1 we have

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(\Omega)} \leq \|\mathbf{u}_r - \mathbf{u}_{r,I}\|_{L^2(\Omega)} + \|\mathbf{u}_s - \mathbf{u}_{s,I}\|_{L^2(\Omega)}.$$

The estimate for the first term

$$\|\mathbf{u}_r - \mathbf{u}_{r,I}\|_{L^2(\Omega)} \leq Ch\|\mathbf{u}_r\|_{H^1(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}$$

is obtained by following the standard arguments.

For the singular part we will estimate the interpolation error on each macroelement Λ_ℓ. Let Λ_ℓ be a fixed macroelement. In what follows we will drop the subindex ℓ, so $\Lambda = \Lambda_\ell$. Suppose that Λ has a singular edge e and a singular vertex v. Let ξ_3 be the unitary vector with the direction of the edge e of Λ, with v as one of its ends, and let ξ_1 and ξ_2 be the unitary vectors with directions of the edges of Λ perpendicular to e, which share with e a vertex other than v (remember that we assume that the singular edge e of Λ is perpendicular to a face of it).

Following Remark 7.3 we can assume that conditions (1) and (2) of Theorem 7.1 hold true for $\mu = 1-\delta$ and $\nu = 1-\beta$, with $\mu \le \nu < 1$, $\beta > \frac{1}{2} - \lambda_v$ and $\delta > 1 - \lambda_e$. For these values of β and δ we know from Theorem 1.1 that

$$\mathbf{u}_s \cdot \xi_i \in V^{1,2}_{\beta,\delta}(\Lambda), \quad i = 1, 2, \qquad \mathbf{u}_s \cdot \xi_3 \in V^{1,2}_{\beta,0}(\Lambda).$$

We will make use also of condition G2 stated in Theorem 7.1.

We observe that for every subdomain $\omega \subset \Lambda$ and for each field $\mathbf{v} \in L^2(\omega)^3$ we have

$$\|\mathbf{v}\|_{L^p(\omega)} \le C \left(\|\mathbf{v}\cdot\xi_1\|_{L^p(\omega)} + \|\mathbf{v}\cdot\xi_2\|_{L^p(\omega)} + \|\mathbf{v}\cdot\xi_3\|_{L^p(\omega)} \right). \tag{44}$$

with C a constant depending only on Λ. For a vectorial function $\mathbf{v}$ we set

$$v_{\xi_i} := \mathbf{v} \cdot \xi_i.$$

By summing on the elements K contained in Λ we have

$$\begin{aligned}
\|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(\Lambda)} &= \sum_{K\subset\Lambda} \|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(K)} \\
&= \sum_{K\subset\Lambda: d(K,v)=0} \|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(K)} + \\
&\qquad \sum_{K\subset\Lambda: \begin{cases} d(K,e)=0 \\ d(K,v)>0 \end{cases}} \|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(K)} + \\
&\qquad \sum_{K\subset\Lambda: \begin{cases} d(K,e)>0 \\ d(K,v)>0 \end{cases}} \|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(K)} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Let us estimate I_2. In this case, the elements K are prisms. We denote $h_{K,i}$ the length of the edge of K with direction ξ_i. We have

$$\begin{aligned}
I_2 &= \sum_{K\subset\Lambda: \begin{cases} d(K,e)=0 \\ d(K,v)>0 \end{cases}} \|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(K)} \\
&\le 2 \sum_{K\subset\Lambda: \begin{cases} d(K,e)=0 \\ d(K,v)>0 \end{cases}} \left(\|\mathbf{u}_s\|^2_{L^2(K)} + \|\mathbf{u}_{s,I}\|^2_{L^2(K)} \right). \qquad (45)
\end{aligned}$$

Let K be an element considered in the sum I_2.Then

$$\begin{aligned}\|\mathbf{u}_s\|^2_{L^2(K)} &\le C\left(\sum_{i=1}^{2}\|R^\nu\theta^\mu R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|^2_{L^2(K)}+\|R^\nu\theta R^{-\nu}\theta^{-1}u_{s,\xi_3}\|^2_{L^2(K)}\right)\\ &\le C\Bigg(\max_{\mathbf{x}\in K}\left[R(\mathbf{x})^\nu\theta(\mathbf{x})^\mu\right]^2\sum_{i=1}^{2}\|R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|^2_{L^2(K)}\\ &\qquad+\max_{\mathbf{x}\in K}\left[R(\mathbf{x})^\nu\theta(\mathbf{x})\right]^2\|R^{-\nu}\theta^{-1}u_{s,\xi_3}\|^2_{L^2(K)}\Bigg).\end{aligned}$$

Since $0\le\theta<1$ and $\mu\le\nu<1$, for all $\mathbf{x}\in K$ with $d(K,e)=0$ we have

$$R(\mathbf{x})^\nu\theta(\mathbf{x})\le R(\mathbf{x})^\nu\theta(\mathbf{x})^\mu\le CR(\mathbf{x})^\mu\theta(\mathbf{x})^\mu=Cr(\mathbf{x})^\mu, \tag{46}$$

and it follows that

$$\|\mathbf{u}_s\|^2_{L^2(K)} \le C\max_{\mathbf{x}\in K}r(\mathbf{x})^{2\mu}\left(\sum_{i=1}^{2}\|R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|^2_{L^2(K)}+\|R^{-\nu}\theta^{-1}u_{s,\xi_3}\|^2_{L^2(K)}\right).$$

Now we remember that if $d(K,e)=0$ then $r(\mathbf{x})\le h_{K,1}\sim h_{K,2}\sim h^{\frac{1}{\mu}}$. Then we have

$$\|\mathbf{u}_s\|^2_{L^2(K)}\le Ch^2\left(\sum_{i=1}^{2}\|R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|^2_{L^2(K)}+\|R^{-\nu}\theta^{-1}u_{s,\xi_3}\|^2_{L^2(K)}\right).$$

Now, by summing over all the corresponding elements we obtain

$$\begin{aligned}\sum_{K\subset\Lambda:\begin{cases}d(K,e)=0\\ d(K,v)>0\end{cases}}\|\mathbf{u}_s\|^2_{L^2(K)} &\le Ch^2\sum_{K\subset\Lambda:\begin{cases}d(K,e)=0\\ d(K,v)>0\end{cases}}\left(\sum_{i=1}^{2}\|R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|^2_{L^2(K)}+\|R^{-\nu}\theta^{-1}u_{s,\xi_3}\|^2_{L^2(K)}\right)\\ &\le Ch^2\left(\|u_{s,\xi_1}\|^2_{V^{1,2}_{\beta,\delta}(\Lambda)}+\|u_{s,\xi_2}\|^2_{V^{1,2}_{\beta,\delta}(\Lambda)}+\|u_{s,\xi_3}\|^2_{V^{1,2}_{\beta,0}(\Lambda)}\right)\end{aligned}\tag{47}$$

since $\beta=1-\nu$ and $\delta=1-\mu$. Next we estimate

$$\sum_{K\subset\Lambda:\begin{cases}d(K,e)=0\\ d(K,v)>0\end{cases}}\|\mathbf{u}_{s,I}\|^2_{L^2(K)}.$$

Given a prismatic element K we have

$$\|\mathbf{u}_{s,I}\|_{L^2(K)}\le C|K|^{-\frac{1}{2}}\|\mathbf{u}_{s,I}\|_{L^1(K)}. \tag{48}$$

We use the stability estimate of Lemma 5.5 to have

$$\|\mathbf{u}_{s,I}\|_{L^1(K)^3}\le C\left(\|\mathbf{u}_s\|_{L^1(K)^3}+\sum_{j=1}^{3}h_j\|\partial_{\xi_j}\mathbf{u}_s\|_{L^1(K)^3}+h_K\|\mathrm{div}\,(u_{1,s},u_{2,s},0)\|_{L^1(K)}\right).$$

Since we are assuming $h_{K,3}\ge h_{K,i}$, $i=1,2$, we can write the previous estimate as

$$\|\mathbf{u}_{s,I}\|_{L^1(K)^3}\le C\left(\|\mathbf{u}_s\|_{L^1(K)^3}+\sum_{j=1}^{3}h_j\|\partial_{\xi_j}\mathbf{u}_s\|_{L^1(K)^3}+h_K\|\mathrm{div}\,\mathbf{u}_s\|_{L^1(K)}\right). \tag{49}$$

Now, for $i=1,2$ we have

$$\begin{aligned}\|u_{s,\xi_i}\|_{L^1(K)} &\le \|R^\nu\theta^\mu\|_{L^2(K)}\|R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|_{L^2(K)}\\ &\le h|K|^{\frac{1}{2}}\|R^{-\nu}\theta^{-\mu}u_{s,\xi_i}\|_{L^2(K)},\end{aligned}\tag{50}$$

and

$$\begin{aligned} \|u_{s,\xi_3}\|_{L^1(K)} &\leq \|R^\nu\theta\|_{L^2(K)}\|R^{-\nu}\theta^{-1}u_{s,\xi_3}\|_{L^2(K)} \\ &\leq h|K|^{\frac{1}{2}}\|R^{-\nu}\theta^{-1}u_{s,\xi_3}\|_{L^2(K)}, \end{aligned} \tag{51}$$

where we used (46) again. Due to property (44), inequalities (50) and (51) allows us to estimate $\|\mathbf{u}_s\|_{L^1(K)}$.

Now we estimate

$$h_{K,j}\|\partial_{\xi_j}u_{s,\xi_i}\|_{L^1(K)}, \qquad i=1,2.$$

We consider firstly the case $j=1,2$. We have

$$h_{K,j}\|\partial_{\xi_j}u_{s,\xi_i}\|_{L^1(K)} \leq Ch|K|^{\frac{1}{2}}\|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j}u_{s,\xi_i}\|_{L^2(K)}. \tag{52}$$

In fact we have

$$h_{K,j}\|\partial_{\xi_j}u_{s,\xi_i}\|_{L^1(K)} \leq h_{K,j}\|R^{\nu-1}\theta^{\mu-1}\|_{L^2(K)}\|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j}u_{s,\xi_i}\|_{L^2(K)}. \tag{53}$$

Since $0\leq 1-\nu\leq 1-\mu<1$, then $R(\mathbf{x})^{1-\nu}\geq R(\mathbf{x})^{1-\mu}$, hence

$$R(\mathbf{x})^{\nu-1}\theta(\mathbf{x})^{\mu-1}\leq R(\mathbf{x})^{\mu-1}\theta(\mathbf{x})^{\mu-1}=r(x)^{\mu-1}. \tag{54}$$

Now, Let $\mathcal{C}_K$ be the part of cylinder $\mathcal{C}_K=\{\mathbf{x}\in A_K\cap B_K : r(\mathbf{x})\leq \max\{h_{K,1},h_{K,2}\}\}$ where A_K is the dihedral angle of K with edge e and B_K is the band defined by the planes containing the top and bottom basis of K. Since $K\subset\mathcal{C}_K$ we have

$$\begin{aligned} \|r^{\mu-1}\|^2_{L^2(K)} \leq \int_{\mathcal{C}_K} r^{2\mu-2}\,d\mathbf{x} = c_K h_{K,3}\int_0^{\max\{h_{K,1},h_{K,2}\}} r^{2\mu-1}\,dr \\ = c_K h_{K,3}\max\{h^\mu_{K,1},h^\mu_{K,2}\}^2 \leq c_K h_{K,3}h^2, \end{aligned}$$

with c_K depending on the angle A_K, and so $c_K\leq C$. Then

$$h_{K,j}\|r^{\mu-1}\|_{L^2(K)} \leq Ch_{K,j}|h_{K,3}|^{\frac{1}{2}}h \leq Ch|K|^{\frac{1}{2}}. \tag{55}$$

In view of (55) and (54), from (53) we have

$$h_{K,j}\|\partial_{\xi_j}u_{s,\xi_i}\|_{L^1(K)} \leq Ch\,|K|^{\frac{1}{2}}\|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j}u_{s,\xi_i}\|_{L^2(K)},$$

which is (52). Now, for $j=3$ (and $i=1,2$) we use that

$$\partial_{\xi_3}\mathbf{u}_{s,\xi_i}=\partial_{\xi_i}\mathbf{u}_{s,\xi_3}\in V^{1,2}_{\beta,0}(\Lambda),$$

as it is easily checked taking into account that $\mathbf{u}_s=\nabla p_s$, with p_s being the singular part of the scalar solution p. Then we have

$$\begin{aligned} h_{K,3}\|\partial_{\xi_3}\mathbf{u}_{s,\xi_i}\|_{L^1(K)} &= h_{K,3}\|\partial_{\xi_i}\mathbf{u}_{s,\xi_3}\|_{L^1(K)} \\ &\leq h_{K,3}|K|^{\frac{1}{2}}\|\partial_{\xi_i}\mathbf{u}_{s,\xi_3}\|_{L^2(K)} \\ &\leq h|K|^{\frac{1}{2}}\|R^{1-\nu}\partial_{\xi_i}\mathbf{u}_{s,\xi_3}\|_{L^2(K)}. \end{aligned} \tag{56}$$

For $i=3$, and $j=1,2$, using that from $0\leq 1-\nu\leq 1-\mu<1$ it follows $R^{\nu-1}\leq R^{\mu-1}\leq r^{\mu-1}$, together with (55), we have

$$\begin{aligned} h_{K,j}\|\partial_{\xi_j}\mathbf{u}_{s,\xi_3}\|_{L^1(K)} &\leq h_{K,j}\|R^{\nu-1}\|_{L^2(K)}\|R^{1-\nu}\partial_{\xi_j}\mathbf{u}_{s,\xi_3}\|_{L^2(K)} \\ &\leq h_{K,j}\|r^{\mu-1}\|_{L^2(K)}\|R^{1-\nu}\partial_{\xi_j}\mathbf{u}_{s,\xi_3}\|_{L^2(K)} \\ &\leq Ch|K|^{\frac{1}{2}}\|R^{1-\nu}\partial_{\xi_j}\mathbf{u}_{s,\xi_3}\|_{L^2(K)}. \end{aligned} \tag{57}$$

Finally, for $i=j=3$ we have

$$\begin{aligned} h_{K,3}\|\partial_{\xi_3}\mathbf{u}_{s,\xi_3}\|_{L^1(K)} &\leq Ch\|R^{1-\nu}\partial_{\xi_3}\mathbf{u}_{s,\xi_3}\|_{L^1(K)} \\ &\leq Ch|K|^{\frac{1}{2}}\|R^{1-\nu}\partial_{\xi_3}\mathbf{u}_{s,\xi_3}\|_{L^2(K)}. \end{aligned} \tag{58}$$

Inserting (50), (51), (52), (56), (57) and (58) into (49), and this into (48), and $h_K \leq h$, we obtain

$$\begin{aligned} \|\mathbf{u}_{s,I}\|_{L^2(K)} \leq\ & Ch\Bigg(\sum_{i=1,2} \|R^{-\nu}\theta^{-\mu} u_{s,\xi_i}\|_{L^2(K)} + \|R^{-\nu}\theta^{-1} u_{s,\xi_3}\|_{L^2(K)} \\ & + \sum_{\substack{i=1,2\\ j=1,2,3}} \|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j} u_{s,\xi_i}\|_{L^2(K)} \\ & + \sum_{j=1,2,3} \|R^{1-\nu}\partial_{\xi_3}\mathbf{u}_{s,\xi_3}\|_{L^2(K)} + \|\operatorname{div}\mathbf{u}_s\|_{L^2(K)}\Bigg). \end{aligned} \tag{59}$$

By squaring and summing these inequalities over all the prismatic elements along the singular edge we obtain

$$\sum_{K\subset\Lambda:\begin{cases} d(K,e)=0\\ d(K,v)>0\end{cases}} \|\mathbf{u}_{s,I}\|^2_{L^2(K)} \leq Ch^2 \left(\|u_{s,\xi_1}\|^2_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_2}\|^2_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_3}\|^2_{V^{1,2}_{\beta,0}(\Lambda)} + \|f\|^2_{L^2(\Omega)}\right), \tag{60}$$

where we used that since $\operatorname{div}\mathbf{u}_s = f - \operatorname{div}\mathbf{u}_r$ it follows

$$\|\operatorname{div}\mathbf{u}_s\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} + \|\operatorname{div}\mathbf{u}_r\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

Therefore, inserting (47) and (60) into (45) we obtain

$$\begin{aligned} I_2 \leq\ & Ch\left(\|u_{s,\xi_1}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_2}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_3}\|_{V^{1,2}_{\beta,0}(\Lambda)} + \|f\|_{L^2(\Omega)}\right) \\ \leq\ & Ch\|f\|_{L^2(\Omega)}. \end{aligned} \tag{61}$$

Now we estimate

$$I_3 = \sum_{K\subset\Lambda:\begin{cases} d(K,e)>0\\ d(K,v)>0\end{cases}} \|\mathbf{u}_s - \mathbf{u}_{s,I}\|^2_{L^2(K)}. \tag{62}$$

In this case, the elements K are anisotropic or isotropic prisms, and isotropic tetrahedra and pyramids. On prisms we have the interpolation error estimate given in Proposition 5.7, which is valid since the prisms K we are considering satisfy $h_{1,K}, h_{2,K} \leq Ch_{3,K}$ due to the fact $\mu \leq \nu$. On the other hand, for pyramids and tetrahedra, we have the error interpolation estimate of Proposition 5.9 which is valid since these elements are assumed isotropic, that is $h_{1,K} \sim h_{2,K} \sim h_{3,K}$ when K is a pyramid or tetrahedron. Summarizing, the interpolation error inequality

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(K)} \leq C\left(\sum_{i=1}^{3} h_{i,K}\|\partial_{\xi_i}\mathbf{u}\|_{L^2(K)} + h_T\|\operatorname{div}\mathbf{u}\|_{L^2(K)}\right). \tag{63}$$

is valid for all the elements elements considered in I_3, where we recall that ξ_3 is a direction parallel to the singular edge and $\xi_1$and ξ_2 are perpendicular to ξ_3.

Furthermore we observe that for all the elements considered in I_3 we have

$$\begin{aligned} h_{i,K} &\leq hd(K,e)^{1-\mu} \leq hr(\mathbf{x})^{1-\mu}, \qquad \forall \mathbf{x}\in K, i=1,2 \\ h_{3,K} &\leq hd(K,v)^{1-\nu} \leq hr(\mathbf{x})^{1-\nu}, \qquad \forall \mathbf{x}\in K. \end{aligned}$$

Now we have, for $j = 1, 2$

$$\begin{aligned}
h_{j,K}^2 \|\partial_{\xi_j} \mathbf{u}_s\|_{L^2(K)}^2 &= h_{j,K}^2 \sum_{i=1}^{3} \|\partial_{\xi_j} \mathbf{u}_{s,\xi_i}\|_{L^2(K)}^2 \\
&\leq h^2 \sum_{i=1}^{3} \|r^{1-\mu} \partial_{\xi_j} \mathbf{u}_{s,\xi_i}\|_{L^2(K)}^2 \\
&\leq h^2 \left(\sum_{i=1}^{2} \|R^{1-\mu}\theta^{1-\mu} \partial_{\xi_j} \mathbf{u}_{s,\xi_i}\|_{L^2(K)}^2 + \|R^{1-\mu} \partial_{\xi_j} \mathbf{u}_{s,\xi_3}\|_{L^2(K)}^2 \right) \\
&\leq h^2 \left(\sum_{i=1}^{2} \|R^{1-\nu}\theta^{1-\mu} \partial_{\xi_j} \mathbf{u}_{s,\xi_i}\|_{L^2(K)}^2 + \|R^{1-\nu} \partial_{\xi_j} \mathbf{u}_{s,\xi_3}\|_{L^2(K)}^2 \right) \\
&= h^2 \left(\sum_{i=1}^{2} \|R^{\beta}\theta^{\delta} \partial_{\xi_j} \mathbf{u}_{s,\xi_i}\|_{L^2(K)}^2 + \|R^{\beta} \partial_{\xi_j} \mathbf{u}_{s,\xi_3}\|_{L^2(K)}^2 \right). \qquad (64)
\end{aligned}$$

Using again that $\partial_{\xi_3} \mathbf{u}_{s,\xi_i} = \partial_{\xi_i} \mathbf{u}_{s,\xi_3}$, we have

$$\begin{aligned}
h_{3,K}^2 \|\partial_{\xi_3} \mathbf{u}_s\|_{L^2(K)}^2 &= h_{3,K}^2 \sum_{i=1}^{3} \|\partial_{\xi_3} \mathbf{u}_{s,\xi_i}\|_{L^2(K)}^2 \\
&= h_{3,K}^2 \sum_{i=1}^{3} \|\partial_{\xi_i} \mathbf{u}_{s,\xi_3}\|_{L^2(K)}^2 \qquad (65) \\
&\leq h^2 \sum_{i=1}^{3} \|R^{1-\nu} \partial_{\xi_i} \mathbf{u}_{s,\xi_3}\|_{L^2(K)}^2 \\
&\leq h^2 \sum_{i=1}^{3} \|R^{\beta} \partial_{\xi_3} \mathbf{u}_{s,\xi_3}\|_{L^2(K)} \qquad (66)
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
h_K \|\operatorname{div} \mathbf{u}_s\|_{L^2(K)} &\leq h_K \left(\|\operatorname{div} \mathbf{u}\|_{L^2(K)} + \|\operatorname{div} \mathbf{u}_r\|_{L^2(K)} \right) \\
&\leq h \left(\|f\|_{L^2(K)} + |\mathbf{u}_r|_{H^1(K)} \right) \qquad (67)
\end{aligned}$$

Now, by inserting (64), (66) and (67) in (63), by summing over the corresponding elements, we obtain from (62)

$$\begin{aligned}
I_3 &\leq Ch \left(\|u_{s,\xi_1}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_2}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_3}\|_{V^{1,2}_{\beta,0}(\Lambda)} + \|f\|_{L^2(\Omega)} + \|\mathbf{u}_r\|_{H^1(\Omega)} \right) \\
&\leq Ch \|f\|_{L^2(\Omega)} \qquad (68)
\end{aligned}$$

Now we deal with I_1. In this case we have to consider only one tetrahedron, which we call T, with an edge on the singular edge (with direction ξ_3) and with the singular vertex v as one of its vertices on the singular edge. We start with

$$\|\mathbf{u}_s - \mathbf{u}_{s,I}\|_{L^2(T)} \leq \|\mathbf{u}_s\|_{L^2(T)} + \|\mathbf{u}_{s,I}\|_{L^2(T)}. \qquad (69)$$

Now

$$\begin{aligned}
\|\mathbf{u}_s\|_{L^2(T)} &= \sum_{i=1}^{2} \|R^\nu \theta^\mu R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \|R^\nu \theta R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)} \\
&\leq \max_{\mathbf{x}\in T}\{R^\nu\theta^\mu\} \sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \max_{\mathbf{x}\in T}\{R^\nu\theta\}\|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)}.
\end{aligned}$$

Since $\theta(\mathbf{x}) \leq C$ and $\mu \leq \nu < 1$ we have

$$R(\mathbf{x})^\nu \theta \leq CR(\mathbf{x})^\nu \theta^\mu \leq CR(\mathbf{x})^\mu \theta^\mu = Cr(\mathbf{x})^\mu,$$

hence

$$\begin{aligned}
\|\mathbf{u}_s\|_{L^2(T)} &\leq C \max_{\mathbf{x}}\{r(\mathbf{x})^\mu\} \left(\sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)} \right) \\
&\leq Ch \left(\sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)} \right) \\
&\leq Ch \left(\|\mathbf{u}_{s,\xi_1}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|\mathbf{u}_{s,\xi_2}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|\mathbf{u}_{s,\xi_3}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} \right) \\
&\leq Ch\|f\|_{L^2(\Omega)}. \qquad (70)
\end{aligned}$$

For the second term in (69) we have, using Proposition 5.8 and that T is isotropic

$$\begin{aligned}
\|\mathbf{u}_{s,I}\|_{L^2(T)} &\leq C|T|^{-\frac{1}{2}}\|\mathbf{u}_{s,I}\|_{L^1(T)} \\
&\leq C|T|^{-\frac{1}{2}} \left(\|\mathbf{u}_s\|_{L^1(T)} + h_T \sum_{j=1}^{3} \|\partial_{\xi_j}\mathbf{u}_s\|_{L^1(T)} \right) \qquad (71)
\end{aligned}$$

We have

$$\begin{aligned}
\|\mathbf{u}_s\|_{L^1(T)} &= \sum_{i=1}^{2} \|R^\nu \theta^\mu R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^1(T)} + \|R^\nu \theta R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^1(T)} \\
&\leq \sum_{i=1}^{2} \|R^\nu\theta^\mu\|_{L^2(K)}\|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \|R^\nu\theta\|_{L^2(T)}\|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)} \\
&\leq \max_{\mathbf{x}\in T}\{R(\mathbf{x})^\nu\theta(\mathbf{x})^\mu\}|T|^{\frac{1}{2}} \sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \\
&\qquad \max_{\mathbf{x}\in T}\{R(\mathbf{x})^\nu\theta(\mathbf{x})\}|T|^{\frac{1}{2}}\|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)}.
\end{aligned}$$

Using (46) we obtain

$$\begin{aligned}
\|\mathbf{u}_s\|_{L^1(T)} &= \max_{\mathbf{x}\in T}\{r(\mathbf{x})^\mu\}|T|^{\frac{1}{2}} \left(\sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)} \right) \\
&\leq Ch|T|^{\frac{1}{2}} \left(\sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}\mathbf{u}_{s,\xi_i}\|_{L^2(T)} + \|R^{-\nu}\theta^{-1}\mathbf{u}_{s,\xi_3}\|_{L^2(T)} \right) \\
&\leq Ch|T|^{\frac{1}{2}} \left(\|u_{s,\xi_1}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_2}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_3}\|_{V^{1,2}_{\beta,0}(\Lambda)} \right) \qquad (72)
\end{aligned}$$

On the other hand, as we did to prove inequalities (52) and (57), we obtain for $j = 1, 2$ and $i = 1, 2$

$$h_T \|\partial_{\xi_j} u_{s,\xi_i}\|_{L^1(K)} \leq Ch|K|^{\frac{1}{2}} \|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j} u_{s,\xi_i}\|_{L^2(K)}, \tag{73}$$

and

$$h_T \|\partial_{\xi_j} u_{s,\xi_3}\|_{L^1(K)} \leq Ch|K|^{\frac{1}{2}} \|R^{1-\nu}\partial_{\xi_j} u_{s,\xi_3}\|_{L^2(K)}. \tag{74}$$

For direction ξ_3 we have

$$h_{3,T}\|\partial_{\xi_3} u_{s,\xi_3}\|_{L^1(T)} \leq h_{3,T}\|R^{\nu-1}\|_{L^2(T)}\|R^{1-\nu}\partial_{\xi_3} u_{s,\xi_3}\|_{L^2(T)}.$$

But, by integration on a sphere centered in v of radius $Ch_{3,T}$, which contains T, we see that

$$h_{3,T}\|R^{\nu-1}\|_{L^2(T)} \leq Ch_{3,T}^{\nu} h_{3,T}^{\frac{3}{2}} \leq Ch|T|^{\frac{1}{2}},$$

where we strongly used the fact that the tetrahedron T is isotropic (or, what we needed is $h_{3,T} \leq h_{1,T}$). This gives

$$h_{3,T}\|\partial_{\xi_3} u_{s,\xi_3}\|_{L^1(T)} \leq Ch|T|^{\frac{1}{2}}\|R^{1-\nu}\partial_{\xi_3} u_{s,\xi_3}\|_{L^2(T)} \leq Ch|T|^{\frac{1}{2}}\|u_{s,\xi_3}\|_{V^{1,2}_{\beta,0}(\Lambda)}. \tag{75}$$

Finally, for $i = 1, 2$,

$$\begin{aligned} h_T\|\partial_{\xi_3} u_{s,\xi_i}\|_{L^1(T)} &= h_T\|\partial_{\xi_i} u_{s,\xi_3}\|_{L^1(T)} \\ &\leq h_T\|\partial_{\xi_i} u_{s,\xi_3}\|_{L^1(T)} \\ &\leq Ch|K|^{\frac{1}{2}}\|R^{1-\nu}\partial_{\xi_j} u_{s,\xi_3}\|_{L^2(K)} \end{aligned} \tag{76}$$

where we used (74) in the last line.

Now we insert (72), (74), (75), and (76) into (71) in order to obtain

$$\|\mathbf{u}_{s,I}\|_{L^2(T)} \leq h\left(\|u_{s,\xi_1}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_2}\|_{V^{1,2}_{\beta,\delta}(\Lambda)} + \|u_{s,\xi_3}\|_{V^{1,2}_{\beta,0}(\Lambda)}\right) \leq Ch\|f\|_{L^2(\Omega)},$$

which together with (70) and (69) give us

$$I_1 \leq Ch\|f\|_{L^2(\Omega)} \tag{77}$$

Equations (77), (61) and (68) conclude the proof of

$$\|\mathbf{u} - \mathbf{u}_I\|_{L^2(\Lambda_\ell)} \leq Ch\|f\|_{L^2(\Omega)}$$

when Λ_ℓ contains a singular vertex and a singular edge. The proofs for macroelements of other types are simpler and so we can skip them. □

In order to apply Theorem 6.1 we define $\mathbf{u}_\pi$ as

$$\mathbf{u}_\pi = \begin{cases} \mathbf{u}_I|_T & \text{if } T \text{ is a prism or a tetrahedron} \\ P_0^T\mathbf{u} & \text{if } T \text{ is a pyramid} \end{cases}$$

Proposition 7.5. *Under the assumptions on the family of meshes $\mathcal{T}_h$ of Theorem 7.4, we have*

$$\|\mathbf{u} - \mathbf{u}_\pi\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}$$

Proof. Let $\mathcal{T}_h^1$ be the set of prisms and tetrahedra of $\mathcal{T}_h$, and $\mathcal{T}_h^2$ be the set of pyramids of $\mathcal{T}_h$. Then we write

$$\|\mathbf{u} - \mathbf{u}_\pi\|^2_{L^2(\Omega)} = \sum_{T\in\mathcal{T}_h^1}\|\mathbf{u} - \mathbf{u}_I\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h^2}\|\mathbf{u} - P_0^T\mathbf{u}\|^2_{L^2(T)} =: J_1 + J_2.$$

By Theorem 7.4 we have

$$J_1 \leq \|\mathbf{u} - \mathbf{u}_I\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.$$

On the other hand, for each pyramid T we have

$$\|\mathbf{u} - P_0^T\mathbf{u}\|_{L^2(T)} \leq Ch_T|\mathbf{u}|_{H^1(T)},$$

with C depending only on the aspect ratio of T, which is uniformly bounded on T and h. Therefore, following the arguments used to estimate I_3 in (68), we obtain

$$J_2 \le Ch\|f\|_{L^2(\Omega)}$$

which concludes the proof. □

Proposition 7.6. *We have*

$$\|p - P_0^{\mathcal{T}_h} p\|_{L^2(\Omega)} \le Ch\|f\|_{L^2(T)}.$$

Proof. We note that for all $T \in \mathcal{T}_h$

$$\|p - P_0^T p\|_{L^2(T)} \le Ch_T |p|_{H^1(T)},$$

and therefore

$$\|p - P_0^{\mathcal{T}_h} p\|_{L^2(\Omega)} \le Ch|p|_{H^1(\Omega)}.$$

The proof concludes using that $\|p\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}$. □

Finally, combining Propositions 7.4, 7.5 and 7.6 with Theorem 6.1 we obtain the next result.

Theorem 7.7. *Under the assumptions of Theorem 7.4 on the family of meshes $\mathcal{T}_h$, we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \quad \le \quad Ch\|f\|_{L^2(\Omega)}.$$

8. Meshes

In this Section we show how a family of meshes with the properties required in the previous Sections can be constructed. We start with the following remark. Suppose that a polyhedral domain Ω contains a singular edge e_S with endpoints A and B, and with $\overset{\circ}{e}_S$ denoting e_S with its ends excluded. We consider a general situation in which Ω may not be a cylinder. Then macroelements Λ such that $\overline{\Lambda} \cap \overset{\circ}{e}_S \neq \emptyset$ can be always taken with a face orthogonal to e_S. More precisely, by choosing a point P on $\overset{\circ}{e}_S$, consider a plane π passing by P and being orthogonal to e_S. Then all the mentioned macroelements can be taken as tetrahedra with a face on π and having AP or BP as an edge. This is illustrated in Figure 3 where a two–dimensional cut of the part of Ω, close to e_S (marked with the thicker line), with a plane passing by e_S is shown. Each large triangle in the figure there corresponds to a tetrahedral macroelement.

In light of Remark 7.2, and in order to simplify the exposition, and focus on the design of the meshes, we will use only one parameter $\mu\,(=\nu)$ for all macroelements with any kind of singularities. Then, let $0 < \mu < 1$ be such that

$$\mu < \lambda_e, \quad \forall \text{ singular edge } e, \qquad \text{and} \qquad \mu < \frac{1}{2} + \lambda_v, \quad \forall \text{ singular vertex } v.$$

In this case, meshes may might become more strongly graded than needed, however, it follows from Remark 7.2 that we do not lose generality. It will be clear from the construction, that meshes with distinct gradings parameters in different regions of the domain can also be defined.

Macroelements are meshed according the following rules. We describe just tetrahedral macroelements as prismatic macroelements have a submesh of cartesian product type.

- Macroelements whose closures contain neither singular vertex nor part of a singular edge are meshed with a uniform tetrahedral grid (see macroelements marked with I in Figure 3).
- Macroelements which do not contain a singular edge, but contain a singular vertex v or an end v of a singular edge, are meshed with a tetrahedral mesh graded toward v (see macroelements marked with II in Figure 3).
- Macroelements containing a singular edge e are meshed with a hybrid mesh graded toward e. These meshes are also graded toward the vertex which is opposite to the face orthogonal to the singular edge (see macroelements marked with III in Figure 3).

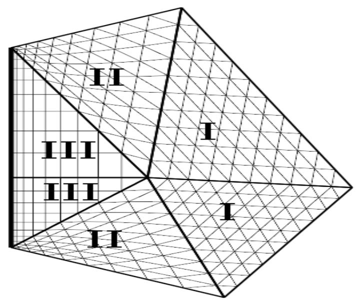

FIGURE 3. 2–d sketch of a possible meshing strategy near a singular edge. The thicker line represents the singular edge e_S.

Figure 3 contains a two–dimensional sketch of a mesh close to a singular edge. In what follows, fixed the parameter $h = \frac{1}{n}$, with $n \in \mathbb{N}$, a mesh $\mathcal{T}_h$ containing $\sim n^3$ elements is described explicitly on each kind of macroelement.

8.1. **Macroelement graded toward an edge.** Let Λ be a tetrahedral macroelement with vertices P_0, P_1, P_2 and P_3. This macroelement has an edge, which we suppose to be P_0P_1, on a singular edge. And it may happen that one of its vertices is a singular one, in this case, we assume that it is P_0. Finally, it is also assumed that the face $P_1P_2P_3$ is orthogonal to P_0P_1. The mesh $\mathcal{T}_h$ on Λ contains tetrahedra, triangular right prisms and pyramids, and it is graded toward the edge P_0P_1 and the vertex P_0, with gradding parameter μ. In what follows we describe all the elements in terms of the barycentric coordinates of the vertices corresponding to the four ordered vertices P_0, P_1, P_2 and P_3 of Λ. Set

$$\gamma = \frac{1}{\mu}.$$

Prisms: for $i, j, l \in \mathbb{N}$ with

$$0 \le l \le n-2 \qquad \text{and} \qquad i+j \le n-l-2$$

consider the prism $p_0p_1p_2p_3p_4p_5$ where $p_s, s = 0, \ldots, 5$ have the barycentric coordinates (the number in row p_s and column P_t is the barycentric coordinate of p_s corresponding to P_t)

	P_0	P_1	P_2	P_3
p_0	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma - \left(\frac{i+j}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$
p_1	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma - \left(\frac{i+j+1}{n}\right)^\gamma$	$\frac{i+1}{n}\left(\frac{i+1+j}{n}\right)^{\gamma-1}$	$\frac{j}{n}\left(\frac{i+1+j}{n}\right)^{\gamma-1}$
p_2	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma - \left(\frac{i+j+1}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{\gamma-1}$	$\frac{j+1}{n}\left(\frac{i+1+j}{n}\right)^{\gamma-1}$
p_3	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	$\left(\frac{n-l-1}{n}\right)^\gamma - \left(\frac{i+j}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$
p_4	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	$\left(\frac{n-l-1}{n}\right)^\gamma - \left(\frac{i+1+j}{n}\right)^\gamma$	$\frac{i+1}{n}\left(\frac{i+1+j}{n}\right)^{\gamma-1}$	$\frac{j}{n}\left(\frac{i+1+j}{n}\right)^{\gamma-1}$
p_5	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	$\left(\frac{n-l-1}{n}\right)^\gamma - \left(\frac{i+j+1}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{\gamma-1}$	$\frac{j+1}{n}\left(\frac{i+1+j}{n}\right)^{\gamma-1}$

and for

$$0 \le l \le n-2, \qquad i \ge 1, \qquad \text{and} \qquad i+j \le n-l-2$$

	P_0	P_1	P_2	P_3
p_0	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma-\left(\frac{i+j}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$
p_1	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma-\left(\frac{i+j+1}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{\gamma-1}$	$\frac{j+1}{n}\left(\frac{i+j+1}{n}\right)^{\gamma-1}$
p_2	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma-\left(\frac{i+j}{n}\right)^\gamma$	$\frac{i-1}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$	$\frac{j+1}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$
p_3	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	$\left(\frac{n-l-1}{n}\right)^\gamma-\left(\frac{i+j}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$
p_4	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	$\left(\frac{n-l-1}{n}\right)^\gamma-\left(\frac{i+j+1}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{\gamma-1}$	$\frac{j+1}{n}\left(\frac{i+j+1}{n}\right)^{\gamma-1}$
p_5	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	$\left(\frac{n-l-1}{n}\right)^\gamma-\left(\frac{i+j}{n}\right)^\gamma$	$\frac{i-1}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$	$\frac{j+1}{n}\left(\frac{i+j}{n}\right)^{\gamma-1}$

Pyramids: for

$$0 \le l \le n-2 \qquad \text{and} \qquad 1 \le i \le n-l-1$$

consider the pyramid $p_0p_1p_2p_3p_4$ with vertices p_s with barycentric coordinates

	P_0	P_1	P_2	P_3
p_0	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma-\left(\frac{n-l-1}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$
p_1	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma-\left(\frac{n-l-1}{n}\right)^\gamma$	$\frac{i-1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$	$\frac{n-l-i}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$
p_2	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	0	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$
p_3	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	0	$\frac{i-1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$	$\frac{n-l-i}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$
p_4	$1-\left(\frac{n-l}{n}\right)^\gamma$	0	$\frac{i}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}$	$\frac{n-l-i}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}$

Tetrahedra: for

$$0 \le l \le n-1 \qquad \text{and} \qquad 1 \le i \le n-l-1$$

consider the tetrahedron $p_0p_1p_2p_3$ with vertices p_s with barycentric coordinates

	P_0	P_1	P_2	P_3
p_0	$1-\left(\frac{n-l}{n}\right)^\gamma$	$\left(\frac{n-l}{n}\right)^\gamma-\left(\frac{n-l-1}{n}\right)^\gamma$	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$
p_1	$1-\left(\frac{n-l-1}{n}\right)^\gamma$	0	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}$
p_2	$1-\left(\frac{n-l}{n}\right)^\gamma$	0	$\frac{i}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}$	$\frac{n-l-i}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}$
p_3	$1-\left(\frac{n-l}{n}\right)^\gamma$	0	$\frac{i+1}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}$

We observe that the projection of the mesh on the plane $P_0P_1P_3$ is the standard two dimensional isotropic mesh graded toward a vertex constructed in [20] (see also [15, Section 8.4]), which shows that prisms satisfies condition G3. Furthermore, following [4, 2] (see also [20, 15]) one can check that this mesh satisfies conditions (42) and (43) with $\nu = \mu$.

8.2. **Macroelement graded only toward a vertex.** We consider again a tetrahedral macroelement Λ with vertices P_0, P_1, P_2 and P_3, assuming that it has to be meshed with grading toward the vertex at P_0. We construct a tetrahedral triangulation of Λ describing the barycentric coordinates of the vertices of each tetrahedral element with respect to P_0, P_1, P_2 and P_3.

Let $p_{i,j,k}$ be the points with barycentric coordinates

$$\begin{aligned}
&\lambda_0 = 1 - \lambda_1 - \lambda_2 - \lambda_3,\\
&\lambda_1 = \frac{i}{n}\left(\frac{i+j+k}{n}\right)^{\gamma-1}, \quad \lambda_2 = \frac{j}{n}\left(\frac{i+j+k}{n}\right)^{\gamma-1}, \quad \lambda_3 = \frac{k}{n}\left(\frac{i+j+k}{n}\right)^{\gamma-1},\\
&0 \le i+j+k \le n.
\end{aligned}$$

Then, the tetrahedra are the ones with vertices

$$\begin{array}{ll}
p_{i,j,k}, p_{i+1,j,k}, p_{i,j+1,k}, p_{i,j,k+1}, & 0 \le i+j+k \le n-1 \\
p_{i+1,j,k}, p_{i,j+1,k}, p_{i,j,k+1}, p_{i+1,j,k+1}, & 0 \le i+j+k \le n-2 \\
p_{i,j+1,k}, p_{i,j,k+1}, p_{i+1,j,k+1}, p_{i,j+1,k+1}, & 0 \le i+j+k \le n-2 \\
p_{i+1,j,k}, p_{i,j+1,k}, p_{i+1,j+1,k}, p_{i+1,j,k+1}, & 0 \le i+j+k \le n-2 \\
p_{i,j+1,k}, p_{i+1,j+1,k}, p_{i+1,j,k+1}, p_{i,j+1,k+1}, & 0 \le i+j+k \le n-2 \\
p_{i+1,j+1,k}, p_{i+1,j,k+1}, p_{i,j+1,k+1}, p_{i+1,j+1,k+1}, & 0 \le i+j+k \le n-3
\end{array}$$

Following [4, 2] (see also [20] and [15, Section 8.4]) one can check that this mesh satisfies condition (43).

8.3. **Macroelements with no singular edges or vertices.** This kind of tetrahedral macroelements are meshed with tetrahedral uniform meshes, which can be described as in Subsection 8.2 with the parameter $\gamma = 1$.

8.4. **Properties of the proposed meshes.** Firstly we note that the meshes of the distinct macroelements described above can be merged in such a way that the resulting global mesh is conforming. This is because the same grading parameters for graded macroelements which share a face can be taken, and on the other hand, since the restrictions of meshes of neighbouring macroelements to the shared face coincide if the grading parameters are suitably taken. In particular, we note that if a face of a tetrahedral macroelement is meshed with rectangles and triangles, this is because an edge of this face (and of the macroelement) is a part of a singular edge. But in this case, if this face is also a face of another macroelement, then the mesh of this second macroelement can be made such that, when restricted to the common face, it consists of the same rectangles and triangles, and this is because this second macroelement has also an edge contained in the same singular edge as the first one.

Proof of Theorem 7.1. Clearly G1 hold. Property G3 and conditions (1) and (2) of the Theorem follow from the developments of Subsections 8.1 and 8.2. It remains to prove property G2: pyramids and tetrahedra of the proposed meshes are isotropic, that is, their aspect ratios are uniformly bounded independently of h. For tetrahedral elements inside macroelements with uniform meshes or meshes graded toward a vertex, the result follows from [4, Section 3]. Then we deal with pyramids appearing in a macroelement graded toward an edge and a vertex, and leave the case of tetrahedra, of the same macroelements, which can be analogously analyzed. Consider a pyramid with vertices $p_0, \dots, p_4$ in a macroelement of vertices P_0, P_1, P_2 and P_3 as in Subsection 8.1. Note that the basis of the pyramid is the parallelogram $p_0p_1p_3p_2$ with

$$p_1 - p_0 = p_3 - p_2 = \frac{1}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}(P_3 - P_2) \tag{78}$$

$$p_2 - p_0 = p_3 - p_1 = \left[\left(\frac{n-l}{n}\right)^{\gamma} - \left(\frac{n-l-1}{n}\right)^{\gamma}\right](P_0 - P_1). \tag{79}$$

So

$$\frac{\gamma}{n}\left(\frac{n-l-1}{n}\right)^{\gamma-1}|P_0 - P_1| \le |p_2 - p_0| = |p_3 - p_1| \le \frac{\gamma}{n}\left(\frac{n-l}{n}\right)^{\gamma-1}|P_0 - P_1|,$$

and

$$\frac{1}{\gamma}\left(\frac{1}{2}\right)^{\gamma-1} \le \frac{1}{\gamma}\left(\frac{n-l-1}{n-l}\right)^{\gamma-1} \le \frac{|p_1 - p_0|}{|p_2 - p_0|} \le \frac{1}{\gamma}.$$

Then the parallelogram $p_0p_1p_3p_2$ is shape-regular since the angle between $P_0 - P_1$ and $P_3 - P_2$ depends only on the macroelement, and so it is away from 0 and π.

Now we prove that there exist constants c_0 and c_1 depending only on γ and the macroelement's vertices such that

$$c_0 \leq \tfrac{|p_4-p_2|}{|p_2-p_0|} \leq c_1 \tag{80}$$

$$c_0 \leq \tfrac{|p_4-p_3|}{|p_2-p_0|} \leq c_1 \quad . \tag{81}$$

After simple computations we obtain

$$\begin{aligned}
p_4 - p_2 &= \left[\left(\frac{n-l}{n}\right)^{\gamma} - \left(\frac{n-l-1}{n}\right)^{\gamma}\right](P_3 - P_0) \\
&\quad + \frac{i}{n}\left[\left(\frac{n-l}{n}\right)^{\gamma-1} - \left(\frac{n-l-1}{n}\right)^{\gamma-1}\right](P_2 - P_3) \\
&= \left[\left(\frac{n-l}{n}\right)^{\gamma} - \left(\frac{n-l-1}{n}\right)^{\gamma}\right]\Bigg\{P_3 - P_0 + \\
&\quad + \frac{\frac{i}{n}\left[\left(\frac{n-l}{n}\right)^{\gamma-1} - \left(\frac{n-l-1}{n}\right)^{\gamma-1}\right]}{\left[\left(\frac{n-l}{n}\right)^{\gamma} - \left(\frac{n-l-1}{n}\right)^{\gamma}\right]}(P_2 - P_3)\Bigg\} \\
&\sim \left[\left(\frac{n-l}{n}\right)^{\gamma} - \left(\frac{n-l-1}{n}\right)^{\gamma}\right](P_3 - P_0) \\
&\sim p_3 - p_1
\end{aligned}$$

where we used that

$$\frac{\frac{i}{n}\left[\left(\frac{n-l}{n}\right)^{\gamma-1} - \left(\frac{n-l-1}{n}\right)^{\gamma-1}\right]}{\left[\left(\frac{n-l}{n}\right)^{\gamma} - \left(\frac{n-l-1}{n}\right)^{\gamma}\right]} \leq \frac{\gamma-1}{\gamma},$$

that the angle between $P_3 - P_0$ and $P_2 - P_3$ is fixed (and depends only on the macroelement) and equation (79). This proves (80), and (81) follows analogously. In order to prove that the pyramid is isotropic now we have to note that the basis $p_0p_1p_3p_2$ is contained in a plane parallel to the one generated by the vectors $P_1 - P_0$ and $P_3 - P_2$, and the face $p_2p_3p_4$ is in a plane which is parallel to the plane $P_0P_2P_3$, and the angle between those planes depends only on the macroelement. □

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