EXPONENTIALLY FITTED DISCONTINUOUS GALERKIN SCHEMES FOR SINGULARLY PERTURBED PROBLEMS

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ABSTRACT. New Discontinuous Galerkin schemes in mixed form are introduced for symmetric elliptic problems of second order. They exhibit reduced connectivity with respect to the standard ones. The modifications in the choice of the approximation spaces and in the stabilization term do not spoil the error estimates. These methods are then used for designing new exponentially fitted schemes for advection dominated equations. The presented numerical tests show the good performances of the proposed schemes.

1. INTRODUCTION

Advection-diffusion problems arise very frequently in applications and it is well known that their numerical discretization requires special care when advection dominates over diffusion. This is the case, for instance, in fluidynamic problems with high Reynolds number, or in semiconductor device simulation under the action of a high electric field.

Here we consider the stationary advection-diffusion model problem

$$\begin{cases} -\mathrm{div}\,(\varepsilon\nabla u - \beta u) = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \\ (\varepsilon\nabla u - \beta u)\cdot \underline{n} = 0 & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \neq \emptyset$, $\underline{n}$ is the unit outward normal vector, and f, g are given functions, with $f \in L^2(\Omega)$, and $g \in H^{1/2}(\Gamma_D)$. Moreover, $\varepsilon = \varepsilon(x)$ and $\beta = \beta(x)$ are given regular functions on $\overline{\Omega}$ with

$$\exists\, \varepsilon_0,\, \varepsilon_M \quad \text{such that} \quad \varepsilon_M \geq \varepsilon(x) \geq \varepsilon_0 > 0. \tag{1.2}$$

We assume that there exists a unique solution of (1.1). This is the case, for instance, if β is a constant, or if there exists $b_0 > 0$ such that div $\beta \geq b_0$.

In the present paper we shall consider the case $\varepsilon \ll |\beta|$ and we shall introduce and analyze new discretization schemes based on Discontinuous Galerkin methods. DG methods for advection–diffusion problems have received a lot of attention in the recent years ([25, 26, 8, 34, 32, 38, 13, 35, 20, 31, 30, 1, 28, 4], e.g.) due to their flexibility in dealing with highly not structured meshes, allowing, for instance, hanging nodes.

Here we consider a stabilizing technique based on the so called *exponential fitting* procedure. Following [17]-[19], and [16], where the stabilization is proposed for semiconductor

2000 *Mathematics Subject Classification.* 65N30.

Key words and phrases. Discontinuous Galerkin Methods, Exponentially Fitted Schemes, Advection-Diffusion Equations.

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device equations in the framework of mixed and hybrid finite element methods, the first step for designing an exponentially fitted stabilization procedure is to symmetrize the equation (1.1) by introducing a new variable ρ, which, in the case of ε being constant and β being the gradient of a scalar function ψ, is defined as

$$\rho = ue^{-\frac{\psi}{\varepsilon}}. \tag{1.3}$$

The general case, which makes use of a local transformation of the type (1.3), similar to the one used in [33] for mixed finite elements, is discussed in details in the second part of section 4. We point out that, in the framework of semiconductor problems, ψ represents the electrostatic potential and ρ is known as Slotboom variable. Equation (1.1) can be rewritten in terms of ρ as

$$\begin{aligned} -\mathrm{div}\,(\varepsilon e^{\frac{\psi}{\varepsilon}}\nabla\rho) &= f \quad \text{in } \Omega,\\ \rho &= \chi \quad \text{on } \Gamma_D,\\ \varepsilon e^{\frac{\psi}{\varepsilon}}\frac{\partial\rho}{\partial n} &= 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{1.4}$$

with $\chi = ge^{-\frac{\psi}{\varepsilon}}$. Then, the symmetric problem (1.4) is approximated by means of suitable Discontinuous Galerkin schemes, and, finally, a discrete version of the change of variable (1.3) is used to substitute ρ_h with the unknown u_h. It turns out that the usual Discontinuous Galerkin schemes ([3], e.g.) are not appropriate for the control of the exponentials which enter the formulation both in volume integrals and in edge integrals (see Remark 4.1). New methods for the symmetric problem are introduced, aiming at reducing the connectivity of the schemes. The reason is twofold. First, when the exponential transformation is applied to the discretization of (1.4), exponentials on one edge interact only with exponentials on the two triangles which contain the edge under consideration. Moreover, the reduced connectivity has the effect to diminish the numerical diffusion. We propose a "recipe" that we apply to several DG schemes of lower order in mixed form. The primal formulation is derived, and error estimates are presented, showing that the reduced schemes keep the same order of convergence as the standard ones.

The paper is organized as follows. In section 2 the notation used throughout the paper are set. Section 3 deals with the symmetric case. Section 4 contains the scheme for advection-dominated problems, first for the simpler case when ε is constant and the flux β is irrotational, and later in the general case. Numerical experiments for the exponentially fitted Interior Penalty scheme are presented in section 5. The proves of the error estimates for the symmetric case are collected in section 6.

2. Notation

We collect in this section the notation used in the rest of the paper for describing the new Discontinuous Galerkin schemes in the general form of [3].

We denote by $\mathcal{T}_h$ the decomposition of the domain Ω into triangles K, by Γ the union of all the edges of $\mathcal{T}_h$, by $\mathcal{E}$ the set of all the edges of $\mathcal{T}_h$ and by $\mathcal{E}_0, \mathcal{E}_D$ and $\mathcal{E}_N$ the sets of the internal edges of $\mathcal{T}_h$, and the edges on the Dirichlet and Neumann boundary, respectively. We also set $\Gamma_0 = \cup\{e : e \in \mathcal{E}_0\}$, $\Gamma_D = \cup\{e : e \in \mathcal{E}_D\}$, $\Gamma_N = \cup\{e : e \in \mathcal{E}_N\}$ and $\Gamma = \Gamma_0 \cup \Gamma_D \cup \Gamma_N$. Given an element $K \in \mathcal{T}_h$, n_K denotes the exterior normal on ∂K, and n is the exterior normal on the boundary of Ω.

As usual, the maximum diameter of the elements in $\mathcal{T}_h$ is h, and we denote by h_K the diameter of an element K. Throughout the paper we assume the mesh to be shape-regular

[24]. We do not make explicit mention to this assumption in the estimates. We will impose additional requirements when necessary.

The space of polynomials of degree less than or equal to k on the set S is $\mathcal{P}_k(S)$, while $\mathcal{P}_k(\mathcal{T}_h)$ is the space of piecewise polynomials on $\mathcal{T}_h$ of degree less than or equal to k.

Let $H^r(\mathcal{T}_h)$ be the space of functions whose restriction to each element K belong to the Sobolev space $H^r(K)$. The space of traces of functions in $H^1(\mathcal{T}_h)$ is contained in $Tr(\Gamma)$, which is defined as $Tr(\Gamma) := \Pi_{K\in\mathcal{T}_h} L^2(\partial K)$. Thus, functions in $Tr(\Gamma)$ are double valued on Γ_0 and single valued on $\partial\Omega$. Given an element K, the restriction of a function v to K is denoted by v_K, even when only the value on ∂K is considered.

For scalar functions $q \in Tr(\Gamma)$ and vector functions $\phi \in Tr(\Gamma)^2$ we introduce the averages $\{q\}$ and $\{\phi\}$, and the jumps $[\![q]\!]$ and $[\![\phi]\!]$, on Γ (using the subscript e to denote their restriction to the edge e). Let e be an interior edge shared by two elements K_1 and K_2, and let n_1 and n_2 be the outward normals to K_1 and K_2, respectively. If $q_i = q_{|\partial K_i}$ then we set

$$\{q\}_e = \frac{1}{2}(q_1 + q_2), \quad [\![q]\!]_e = q_1 n_1 + q_2 n_2, \qquad \text{on } e \subset \Gamma_0.$$

We define ϕ_1 and ϕ_2 analogously and we set

$$\{\phi\}_e = \frac{1}{2}(\phi_1 + \phi_2), \quad [\![\phi]\!]_e = \phi_1 \cdot n_1 + \phi_2 \cdot n_2, \qquad \text{on } e \subset \Gamma_0.$$

Notice that these definitions do not depend on assigning an ordering to the elements K_1 and K_2. Also note that the jump of a scalar function is a vector parallel to the normal, and the jump of a vector function is a scalar quantity. On boundary edges we set

$$\{q\}_e = q, \quad [\![q]\!]_e = qn, \quad \{\phi\}_e = \phi, \quad [\![\phi]\!]_e = \phi \cdot n \quad \text{on } e \subset \partial\Omega,$$

where n is the exterior normal of Ω.

We also define the projection $\Pi_h : H^1(\mathcal{T}_h) \to Tr(\Gamma)$ as

$$\Pi_h v_{|\partial K} = \prod_{e\subseteq\partial K} \Pi_0^{e,K} v,$$

where the product in the right hand side is taken on the edges e of K, and $\Pi_0^{e,K} v$ is the L^2-projection of the trace on e of v_K on the space $\mathcal{P}_0(e)$ of constant functions on e (explicit dependence on K will not be specified, if it is evident from the context, or when the projection is single-valued on e, for instance when $v \in H^1(\Omega)$).

For a function $v \in H^1(\mathcal{T}_h)$, $\nabla_h v$ denotes the function in $L^2(\Omega)$ given by $(\nabla_h v)_K = \nabla v_K$, and for a vector $\tau \in [H^1(\mathcal{T}_h)]^2$, $\operatorname{div}_h \tau$ denotes the function in $L^2(\Omega)$ defined by $(\operatorname{div}_h \tau)_K = \operatorname{div} \tau_K$. Moreover, we define the semi-norm $|v|^2_{1,h} = \sum_{K\in\mathcal{T}_h} \|\nabla v_K\|^2_{0,K}$.

Finally, we introduce the following notation that will be used in section 4. Let $\mathcal{C}(\mathcal{T}_h) = \prod_{K\in\mathcal{T}_h} \mathcal{C}(K)$, where $\mathcal{C}(K)$ denotes the set of continuous functions on K. We set $v_{m,K} := \min\{v_K(x) : x \in K\}$ and if e is an edge of K we set $v_{m,e,K} := \min\{v_K(s) : s \in e\}$, and analogously we denote by $v_{M,K}$ and $v_{M,e,K}$ the maximum on K and on $e \subseteq \partial K$.

We shall also use the standard notation for the mean value of a function f, that is

$$⨍_e f\, ds = \frac{1}{|e|}\int_e f\, ds, \qquad ⨍_K f\, dx = \frac{1}{|K|}\int_K f\, dx$$

on edges e and elements K.

3. The symmetric problem

We introduce a family of Discontinuous Galerkin schemes for a symmetric elliptic problem that here we write in mixed form

$$
\begin{aligned}
a^{-1}\sigma &= \nabla\rho \quad \text{in } \Omega\\
-\operatorname{div}\sigma &= f \quad \text{in } \Omega\\
\rho &= \chi \quad \text{on } \Gamma_D\\
\sigma\cdot n &= 0 \quad \text{on } \Gamma_N
\end{aligned}
\tag{3.1}
$$

with $a \in L^\infty(\Omega)$ satisfying $a \geq a_0 > 0$ on the domain Ω. When $a = \varepsilon e^{\frac{\psi}{\varepsilon}}$, (3.1) is the mixed form of (1.4).

Let us define the following finite element spaces

$$V_h = \mathcal{P}_1(\mathcal{T}_h), \qquad \Sigma_h = \mathcal{P}_0(\mathcal{T}_h)^2.$$

We introduce the Discontinuous Galerkin (DG) schemes in the general formulation of [3] as follows: Find $\sigma_h \in \Sigma_h$ and $\rho_h \in V_h$ such that for all $\tau \in \Sigma_h$ and $v \in V_h$ we have

$$
\begin{aligned}
&\int_\Omega a_h^{-1}\sigma_h\cdot\tau\,dx + \int_\Omega \rho_h \operatorname{div}_h\tau\,dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{\rho}\tau\cdot n_K\,ds = 0,\\
&\int_\Omega \sigma_h\cdot\nabla_h v\,dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{\sigma}\cdot n_K v\,ds = \int_\Omega f v\,dx,
\end{aligned}
\tag{3.2}
$$

where $a_h \in \mathcal{P}_0(\mathcal{T}_h)$ is a piecewise constant approximation of a (for instance, the L^2-projection, or the approximation defined in (4.10)), and $\widehat{\rho}$ and $\widehat{\sigma}$ are the numerical fluxes that will be defined in the sequel and that will identify the different schemes.

We slightly weaken the penalization in the definition of the fluxes introduced in [3]. For the schemes under consideration, the scalar flux $\widehat{\rho}$ is kept unchanged, but we take $\widehat{\sigma} \in \mathcal{P}_0(\mathcal{E})^2$ and replace $[\![\,\rho_h\,]\!]$ in $\widehat{\sigma}$ with $[\![\,\Pi_h\rho_h\,]\!]$.

3.1. **Description of the methods.** Now we write the explicit form of the fluxes for the modified schemes corresponding to Interior Penalty and Local Discontinuous Galerkin methods.

In the following, χ_h denotes a piecewise constant approximation of χ on Γ_D.

Starting from the Interior Penalty (IP) [2, 29, 37] we choose

$$
\widehat{\rho}(\rho_h)_{|e} = \begin{cases} \{\rho_h\} & \text{if } e\in\mathcal{E}_0\cup\mathcal{E}_N\\ \chi_h & \text{if } e\in\mathcal{E}_D\end{cases}
\tag{3.3}
$$

$$
\widehat{\sigma}(\rho_h)_{|e} = \begin{cases} \{a_h\nabla_h\rho_h\} - \mu\,[\![\,\Pi_h\rho_h\,]\!] & \text{if } e\in\mathcal{E}_0\\ a_h\nabla_h\rho_h - \mu(\Pi_h\rho_h - \chi_h)n & \text{if } e\in\mathcal{E}_D\\ 0 & \text{if } e\in\mathcal{E}_N.\end{cases}
\tag{3.4}
$$

The modified Local Discontinuous Galerkin (LDG) [27] is obtained by choosing

$$
\widehat{\rho}(\rho_h)_{|e} = \begin{cases} \{\rho_h\} - \theta\cdot[\![\,\rho_h\,]\!] & \text{if } e\in\mathcal{E}^0\\ \chi_h & \text{if } e\subset\mathcal{E}_D\\ \rho_h & \text{if } e\subset\mathcal{E}_N\end{cases}
\tag{3.5}
$$

$$
\widehat{\sigma}(\rho_h,\sigma_h)_{|e} = \begin{cases} \{\sigma_h\} + \theta\,[\![\,\sigma_h\,]\!] - \mu\,[\![\,\Pi_h\rho_h\,]\!] & \text{if } e\in\mathcal{E}^0\\ \sigma_h - \mu(\Pi_h\rho_h - \chi_h)n & \text{if } e\subset\mathcal{E}_D\ .\\ 0 & \text{if } e\subset\mathcal{E}_N\end{cases}
\tag{3.6}
$$

The penalty function μ depends on a_h and in both cases, IP and LDG, is taken on each edge e as

$$\mu_e = \eta_e \frac{\{a_h\}_e}{h_e} \tag{3.7}$$

where h_e is the length of e and η_e is a constant on the edge e, uniformly bounded below as we shall specify later on. The vector θ in LDG is also constant on each edge e and it is independent of a_h.

Remark 3.1. The same "recipe" (i.e. choosing the vector variable space Σ_h made of piecewise constants and introducing the projection Π_h in the penalization term) could be applied to other schemes of DG type. For instance, a modified NIPG [36] could be introduced and studied in a similar way as done here for modified IP and LDG. We mention that in [10, 11] the projection Π_h is introduced in the penalization term for an over-penalized NIPG scheme, with the effect of allowing the use of a simple block preconditioner that keep the condition number at the order $O(h^{-2})$. Instead, when the modification is applied to Bassi et al [7] (and to Brezzi et al. [15], resp.) we do not introduce a new scheme, but the modified IP scheme (3.3)-(3.4) (and the modified LDG scheme (3.5)-(3.6), resp.) is recovered with special choices of the penalization parameter μ (and θ). Therefore, in the remaining part of the paper only the modified IP and LDG schemes will be explicitly presented and discussed.

We also mention that a weak penalization with the projection in the jump has been used in [5, 6] for a family of IP schemes (symmetric and non-symmetric) given in primal form.

In order to establish the wellposedness and some convergence properties of the schemes just described, we will make use of their primal formulations, which we introduce next.

Following [3] and taking into account that a_h is piecewise constant, and therefore $a_h \nabla_h V_h \subset \Sigma_h$, we can eliminate the vector variable σ_h from (3.2) and obtain the primal formulation of each scheme in the scalar variable ρ_h. With the same notation as in equation (3.9) of [3], we have

$$\sigma_h = \sigma_h(\rho_h) := a_h \nabla_h \rho_h - a_h r(\llbracket \widehat{\rho}(\rho_h) - \rho_h \rrbracket) - a_h l(\{\widehat{\rho}(\rho_h) - \rho_h\}), \tag{3.8}$$

where $r : [L^2(\Gamma)]^2 \to \Sigma_h$ and $l : L^2(\Gamma_0) \to \Sigma_h$ are the lifting operators defined by

$$\int_\Omega r(\phi) \cdot \tau \, dx = -\int_\Gamma \phi \cdot \{\tau\} \, ds, \qquad \int_\Omega l(q) \cdot \tau \, dx = -\int_{\Gamma_0} q \llbracket \tau \rrbracket \, ds, \quad \forall \tau \in \Sigma_h. \tag{3.9}$$

Then, we obtain the following primal formulation of problem (3.2): Find $\rho_h \in V_h$, such that for all $v \in V_h$

$$\begin{aligned} \int_\Omega a_h \nabla_h \rho_h \cdot \nabla_h v \, dx + \int_\Gamma (\llbracket \widehat{\rho} - \rho_h \rrbracket \cdot \{a_h \nabla_h v\} - \{\widehat{\sigma}\} \cdot \llbracket v \rrbracket) \, ds \\ + \int_{\Gamma_0} (\{\widehat{\rho} - \rho_h\} \llbracket a_h \nabla_h v \rrbracket - \llbracket \widehat{\sigma} \rrbracket \cdot \{v\}) \, ds = \int_\Omega f v \, dx. \end{aligned} \tag{3.10}$$

The primal formulation of the modified methods is obtained using in (3.10) the definition of the fluxes.

Consider first the modified IP method. Using definition (3.3)-(3.4) of the numerical fluxes, equation (3.10) becomes

$$\int_\Omega a_h \nabla_h \rho_h \cdot \nabla_h v\, dx - \int_{\Gamma_0 \cup \Gamma_D} (\llbracket \rho_h \rrbracket \cdot \{a_h \nabla_h v\} + \llbracket v \rrbracket \cdot (\{a_h \nabla_h \rho_h\} - \mu \llbracket \Pi_h \rho_h \rrbracket))\, ds + \int_{\Gamma_D} (\chi_h n \cdot \{a_h \nabla_h v\} - \llbracket v \rrbracket \cdot \mu \Pi_h \chi_h n)\; ds = \int_\Omega f v\, dx.$$

We have

$$\int_\Gamma \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket v \rrbracket\; ds = \int_\Gamma \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket\; ds.$$

Therefore, (3.10) takes the form

$$B_h(\rho_h, v) = \int_\Omega f v\, dx - \int_{\Gamma_D} (\chi_h n \cdot a_h \nabla_h v - \mu \chi_h v)\; ds \qquad \forall v \in V_h, \tag{3.11}$$

with the bilinear form $B_h = B_h^{IP} : H^2(\mathcal{T}_h)^2 \to \mathbb{R}$ given by

$$\begin{aligned} B_h^{IP}(\rho_h, v) = & \int_\Omega a_h \nabla_h \rho_h \cdot \nabla_h v\, dx - \int_{\Gamma_0 \cup \Gamma_D} \llbracket \rho_h \rrbracket \cdot \{a_h \nabla_h v\}\, ds \\ & - \int_{\Gamma_0 \cup \Gamma_D} \llbracket v \rrbracket \cdot \{a_h \nabla_h \rho_h\}\, ds + \int_{\Gamma_0 \cup \Gamma_D} \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket\; ds. \end{aligned} \tag{3.12}$$

Notice that in this case, formulation (3.11)-(3.12) is equivalent to using the midpoint quadrature formula for the edge integrals in the primal formulation of the standard Interior Penalty method.

Now consider the modified LDG method. From definition (3.5)-(3.6) we have

$$\llbracket \widehat{\rho} - \rho_h \rrbracket = \begin{cases} -\llbracket \rho_h \rrbracket & \text{if } e \in \mathcal{E}_0 \\ (\chi_h - \rho_h) n & \text{if } e \in \mathcal{E}_D \\ 0 & \text{if } e \in \mathcal{E}_N \end{cases}, \qquad \{\widehat{\rho} - \rho_h\} = -\theta \cdot \llbracket \rho_h \rrbracket \quad \text{if } e \in \mathcal{E}_0.$$

Then, equation (3.10) takes the form (3.11) with $B_h = B_h^{LDG} : H^2(\mathcal{T}_h)^2 \to \mathbb{R}$ given by

$$\begin{aligned} B_h^{LDG}(\rho_h, v) = & \int_\Omega a_h \nabla_h \rho_h \cdot \nabla_h v\, dx - \int_{\Gamma_0 \cup \Gamma_D} (\llbracket \rho_h \rrbracket \cdot \{a_h \nabla_h v\} + \{a_h \nabla_h \rho_h\} \cdot \llbracket v \rrbracket)\, ds \\ & - \int_{\Gamma_0} (\llbracket a_h \nabla_h \rho_h \rrbracket \theta \cdot \llbracket v \rrbracket + \llbracket \rho_h \rrbracket \cdot \theta \llbracket a_h \nabla_h v \rrbracket)\, ds \\ & + \int_{\Gamma_0 \cup \Gamma_D} \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket\; ds + \int_{\Gamma_0 \cup \Gamma_D} \{\Upsilon\} \cdot \llbracket v \rrbracket\; ds + \int_{\Gamma_0} \theta \llbracket \Upsilon \rrbracket \cdot \llbracket v \rrbracket\; ds, \end{aligned} \tag{3.13}$$

with

$$\Upsilon = a_h r(\llbracket \widehat{\rho} - \rho_h \rrbracket) - a_h l(\theta \cdot \llbracket \rho_h \rrbracket).$$

Notice that in the simplest case of homogeneous Dirichlet boundary conditions on the entire $\partial\Omega$, we have

$$\begin{aligned} \int_{\Gamma_0 \cup \Gamma_D} \{\Upsilon\} \cdot \llbracket v \rrbracket\; ds + \int_{\Gamma_0} \theta \llbracket \Upsilon \rrbracket \cdot \llbracket v \rrbracket\; ds &= - \int_\Omega \Upsilon \cdot (r(\llbracket v \rrbracket) + l(\theta \cdot \llbracket v \rrbracket))\, dx \\ &= \int_\Omega a_h (r(\llbracket \rho_h \rrbracket) + l(\theta \cdot \llbracket \rho_h \rrbracket)) \cdot (r(\llbracket v \rrbracket) + l(\theta \cdot \llbracket v \rrbracket))\, dx. \end{aligned}$$

We note that the presence of the lifting operators introduces in form (3.13) an explicit dependence on the space Σ_h. Therefore, a direct link (via a quadrature formula) between (3.13) and the primal form of the standard LDG method is not possible.

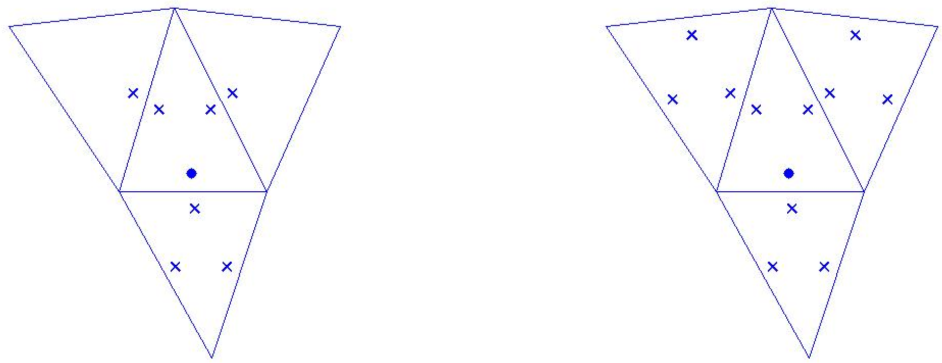

FIGURE 1. Connectivity for modified IP (left) and standard IP (right).

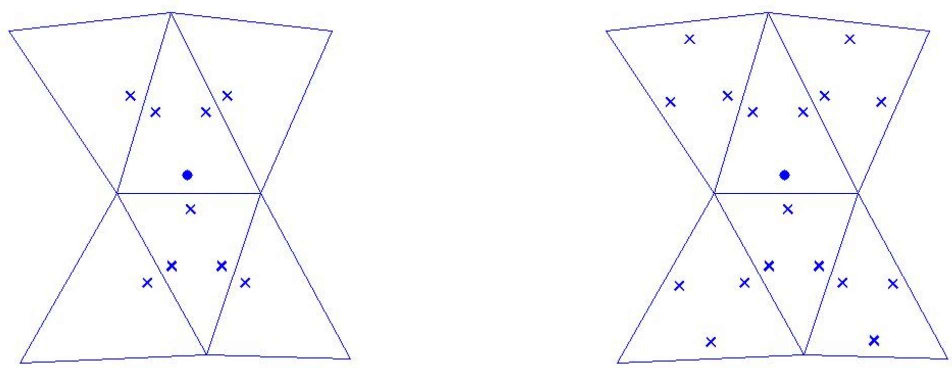

FIGURE 2. Left: Connectivity for modified LDG (left) standard LDG (right).

Remark 3.2. We point out that the modified schemes just defined have a reduced connectivity compared with standard IP and LDG methods, as illustrated in Figures 1 and 2. As it will be done in section 4, the degrees of freedom are taken in the midpoints of the edges, and the *connected degrees of freedom* to a given node (visualized with a $\bullet$) are marked by $\times$.

3.2. **Analysis of the methods.** Now we deal with the wellposedness of our schemes, by proving in the next lemma that the bilinear forms B_h^{IP} and B_h^{LDG} are coercive in $V_h \times V_h$ with respect to a convenient norm.

For a function $v \in H^1(\mathcal{T}_h)$, let $|v|_*$ be the seminorm defined by

$$|v|_*^2 = \sum_{e \in \mathcal{E}_0 \cup \mathcal{E}_D} \frac{\{a_h\}_e}{h_e} \, \| \llbracket \Pi_h v \rrbracket \|_{0,e}^2 \, . \tag{3.14}$$

Then, we define the norm $|||\cdot|||_h$ for functions $v \in H^2(\mathcal{T}_h)$ by

$$|||v|||_h^2 = |\sqrt{a_h}\, v|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\sqrt{a_h} v|_{2,K}^2 + |v|_*^2,$$

which is actually a norm in $H^2(\mathcal{T}_h)$ since $a_h \geq a_0 > 0$ and $\Gamma_D \neq \emptyset$.

In the proof of the next lemma we consider for simplicity the case of homogeneous Dirichlet boundary conditions, so $\Gamma_N = \emptyset$, $\Gamma_D = \partial\Omega$ and $\chi_h = 0$ on $\partial\Omega$.

Lemma 3.1. *There exists a constant $\eta_0 > 0$ depending only on the minimum angle α_h of the mesh $\mathcal{T}_h$ and on the parameter θ for the LDG method, such that, for μ defined in (3.7) with $\eta_e \geq \eta_0$ for all $e \in \mathcal{E}$, we have*

$$B_h(v,v) \geq C_s |||v|||_h^2 \qquad \forall v \in V_h \tag{3.15}$$

with the constant C_s independent of h, and where B_h is either B_h^{IP} or B_h^{LDG}.

Proof. We have

$$B_h(v,v) = \int_\Omega a_h |\nabla_h v|^2 \, dx - 2\int_\Gamma [\![v]\!] \cdot \{a_h \nabla_h v\} \, ds + \int_\Gamma \mu [\![\Pi_h v]\!] \cdot [\![\Pi_h v]\!] \, ds + \Xi,$$

where for the IP method $\Xi = 0$, while for LDG we have

$$\Xi = -2\int_{\Gamma_0} [\![a_h \nabla_h v]\!] \theta \cdot [\![v]\!] + \textit{a nonnegative term.}$$

Taking into account that a_h is piecewise constant, we obtain for $v \in V_h$

$$\begin{aligned}\left|\int_\Gamma [\![v]\!] \cdot \{a_h \nabla v\} \, ds\right| &= \left|\int_\Gamma [\![\Pi_h v]\!] \cdot \{a_h \nabla v\} \, ds\right| \\ &\leq \left(2\sum_{e\in\mathcal{E}} \frac{a_{h,e}}{h_e} \| [\![\Pi_h v]\!] \|_{0,e}^2\right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}} h_e \|\{\sqrt{a_h}\nabla v\}\|_{0,e}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now

$$\begin{aligned}\sum_{e\in\mathcal{E}} h_e \|\{\sqrt{a_h}\nabla v\}\|_{0,e}^2 &\leq C(\alpha_h) \sum_{e\in\mathcal{E}} \sum_{e\subset T} \|\sqrt{a_h}\nabla v\|_{0,T}^2 \\ &\leq C(\alpha_h) |\sqrt{a_h} v|_{1,h}^2.\end{aligned}$$

Therefore

$$\left|\int_\Gamma [\![v]\!] \cdot \{a_h \nabla v\} \, ds\right| \leq C(\alpha_h) |\sqrt{a_h} v|_{1,h} |v|_*.$$

In a similar way, we have

$$\left|\int_{\Gamma_0} [\![a_h \nabla_h v]\!] \theta \cdot [\![v]\!]\right| \leq C(\alpha_h,\theta) |\sqrt{a_h} v|_{1,h} |v|_*.$$

Thus, for both methods considered we have

$$B_h(v,v) \geq |\sqrt{a_h} v|_{1,h}^2 + \eta |v|_*^2 - C(\alpha_h,\theta) |\sqrt{a_h} v|_{1,h} |v|_*.$$

Using the arithmetic-geometric inequality with $\lambda > 0$ we have

$$C(\alpha_h,\theta) |\sqrt{a_h} v|_{1,h} |v|_* \leq C(\alpha_h,\theta) \frac{\lambda}{2} |\sqrt{a_h} v|_{1,h}^2 + C(\alpha_h,\theta) \frac{1}{2\lambda} |v|_*^2$$

Taking $\lambda = \frac{1}{C(\alpha_h,\theta)}$, we have

$$B_h(v,v) \geq \frac{1}{2} |\sqrt{a_h} v|_{1,h}^2 + \left(\eta - \frac{C(\alpha_h,\theta)^2}{2}\right) |v|_*^2,$$

which gives (3.15) for $\eta \geq \eta_0$ with

$$\eta_0 = \frac{1}{2} + \frac{C(\alpha_h,\theta)^2}{2}.$$

and $C_s = \frac{1}{2}$. □

Lemma 3.1 implies immediately existence and uniqueness of the solution of problem (3.11). Then, existence and uniqueness of σ_h follows from (3.8).

The next result contains error estimates for both methods, when $\rho \in H^2(\Omega)$. We postpone its proof to section 6.

Theorem 3.2. *Let ρ_h and σ_h be the approximate solutions obtained using the modified IP or modified LDG schemes with the penalization parameter μ as in (3.7) with $\eta_e \geq \eta_0$ for all $e \in \mathcal{E}$, as stated in Lemma 3.1. Then, we have*

$$|||\rho - \rho_h|||_h \leq Ch|\rho|_{2,\Omega}$$
$$\|\sigma - \sigma_h\|_{0,\Omega} \leq Ch|\rho|_{2,\Omega}$$

where the constant C depends on η_0 and on the regularity of the mesh, and on θ for LDG, but it is independent on h and ρ.

Similarly to the standard LDG case, wellposedness of the modified LDG scheme is still verified when the penalization parameter μ is taken as

$$\mu = \eta_e\{a_h\} \qquad \text{on } e \in \mathcal{E}, \tag{3.16}$$

rather than as (3.7). Here η_e is a positive constant for each edge e [22, 23]. Indeed, the mixed formulation (3.2) with the LDG numerical fluxes (3.5)-(3.6) takes the form: Find $(\rho_h, \sigma_h) \in V_h \times \Sigma_h$ such that for all $(v, \tau) \in V_h \times \Sigma_h$

$$\int_\Omega a_h^{-1}\sigma_h \cdot \tau \, dx + b(\rho_h, \tau) = \int_{\Gamma_D} \chi_h \, \tau \cdot n \, ds \tag{3.17}$$

$$-b(v, \sigma_h) + \int_{\Gamma_0 \cup \Gamma_D} \mu \, [\![\Pi_h \rho_h]\!] \cdot [\![\Pi_h v]\!] \, ds = \int_\Omega f v \, dx + \int_{\Gamma_D} \mu \Pi_h \chi_h \, v \, ds, \tag{3.18}$$

where b is the bilinear form defined by

$$b(v, \tau) = \int_\Omega v \mathrm{div}_h \tau \, dx - \int_{\Gamma_0} (\{v\} [\![\tau]\!] - \theta \cdot [\![v]\!] [\![\tau]\!]) \, ds - \int_{\Gamma_N} v \, \tau \cdot n \, ds. \tag{3.19}$$

We check uniqueness. In fact, if $f = 0$ and $\chi_h = 0$, taking $v = \rho_h$ in (3.18) and $\tau = \sigma_h$ in (3.17), and adding the resulting equations, we obtain

$$\int_\Omega a_h^{-1}\sigma_h \cdot \sigma_h dx + \int_{\Gamma_0 \cup \Gamma_D} \mu \, [\![\Pi_h \rho_h]\!] \cdot [\![\Pi_h \rho_h]\!] \, ds = 0.$$

This implies that $\sigma_h = 0$ in Ω and that ρ_h is continuous at the midpoints of the interelements and vanishes at the midpoints of the Dirichlet edges. Thus, (3.17) gives $b(\rho_h, \tau) = 0$ for all $\tau \in \Sigma_h$. It follows that $\int_\Omega \nabla_h \rho_h \cdot \tau = 0$ for all $\tau \in \Sigma_h$, and therefore $\nabla_h \rho_h$ vanishes in Ω. Since $\Gamma_D \neq \emptyset$, ρ_h must be identically zero. So, we proved the following Lemma.

Lemma 3.3. *If the penalization parameter μ is taken as in (3.16) with $\eta_e > 0$ for all $e \in \mathcal{E}$, then the modified LDG method defines a unique solution $(\rho_h, \sigma_h) \in V_h \times \Sigma_h$.*

In order to obtain error estimates, it is convenient to introduce, for $v \in H^1(\mathcal{T}_h)$, the seminorm

$$|v|_\star = \left(\sum_{e \in \mathcal{E}} \eta_e \|\Pi_h v\|_{0,e}^2 \right)^{\frac{1}{2}} . \tag{3.20}$$

The reader should not confuse it with the seminorm $|\cdot|_*$ defined in (3.14).

Theorem 3.4. *Under the same conditions of Lemma 3.3, we have the error estimate*

$$\|\sigma - \sigma_h\|_{0,\Omega} + |\rho - \rho_h|_\star \leq Ch^{\frac{1}{2}}|\rho|_{2,\Omega}.$$

Finally, we report the L^2–error estimates for the methods considered here.

Theorem 3.5. *For the modified IP or LDG schemes with μ defined by (3.7), we have*

$$\|\rho - \rho_h\|_{0,\Omega} \leq Ch^2|\rho|_{2,\Omega}. \tag{3.21}$$

For the modified LDG with μ defined by (3.16), and assuming a quasiuniform mesh, we have

$$\|\rho - \rho_h\|_{0,\Omega} \leq Ch|\rho|_{2,\Omega}. \tag{3.22}$$

Notice that we lose half power of h in the error estimate for σ, and one power of h in the L^2 error estimate for ρ when the parameter μ is taken as in (3.16). However, the suboptimality of the convergence order is in accordance with similar results for the standard LDG scheme (see [21] and Table 1 of [23]).

The proofs of Theorems 3.4 and 3.5 are given in section 6.

4. Exponentially Fitted DG schemes

We introduce first the exponentially fitted DG scheme using the global transformation (1.3). In the second part, we shall consider the more general case $\varepsilon(x)$ and $\beta(x)$ piecewise constant, when (1.3) exists only locally.

4.1. The case $\beta = \nabla\psi$ with ψ continuous.

Assume that the function $\varepsilon(x)$ is constant on Ω and that there exists a potential $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\beta = \nabla\psi$. Equation (1.1) can be written in the symmetric form (1.4) and the schemes introduced in section 3 can be applied with $a = \varepsilon e^{\frac{\psi}{\varepsilon}}$. Since we are interested in approximating the variable u, we need to introduce a discrete analogue of the application $u \to \rho$ defined by (1.3). Given a triangle $K \in \mathcal{T}_h$, we denote by e^i, $i = 1, 2, 3$ the three edges of K. In $\mathcal{P}_1(K)$ we choose the basis function φ_K^i, $i = 1, 2, 3$, as the polynomial of degree 1 which takes the value 1 in the midpoint of the edge e^i and the value 0 in the midpoint of the other two edges. For $v \in \mathcal{P}_1(K)$, we write

$$v_K(x) = v^1\varphi_K^1(x) + v^2\varphi_K^2(x) + v^3\varphi_K^3(x).$$

Moreover, for $K \in \mathcal{T}_h$ and e edge of K we define

$$E(K, e) = \fint_e e^{-\frac{\psi}{\varepsilon}|_K}\, ds, \tag{4.1}$$

where we used the notation introduced at the end of Section 2 for the mean value. Finally, we introduce the operator $T : V_h \to V_h$ such that, for all $v \in V_h$

$$(Tv)_K = E(K, e^1)v^1\varphi_K^1 + E(K, e^2)v^2\varphi_K^2 + E(K, e^3)v^3\varphi_K^3. \tag{4.2}$$

The discrete analogue of (1.3) is then defined by

$$\rho_h = Tu_h. \tag{4.3}$$

Similarly, we define the boundary value χ_h, approximation of χ in (1.4), as

$$\chi_{h|e} := E(K, e)g_{h|e}, \qquad \forall e \in \mathcal{E}_D, \tag{4.4}$$

g_h being the piecewise constant approximation of g defined by $g_{h|e} = \Pi_0^e g$.

Then, by replacing ρ_h with Tu_h in formulation (3.2) and in the definitions of the numerical fluxes (3.3) and (3.4) for the modified IP scheme, and of the numerical fluxes (3.5) and (3.6) for the modified LDG scheme, we obtain the exponentially fitted IP (and LDG, resp.) scheme for the variables σ_h and u_h: Find $\sigma_h \in \Sigma_h$ and $u_h \in V_h$ such that, for all $\tau \in \Sigma_h$ and $v \in V_h$, it holds

$$\begin{aligned} &\int_\Omega a_h^{-1}\sigma_h \cdot \tau \, dx + \int_\Omega Tu_h \mathrm{div}_h \, \tau \, dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \widehat{\rho}\tau \cdot n_K \, ds = 0 \\ &\int_\Omega \sigma_h \cdot \nabla_h v \, dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \widehat{\sigma} \cdot n_K v \, ds = \int_\Omega fv \, dx \, . \end{aligned} \tag{4.5}$$

For the IP scheme, the numerical fluxes are given by

$$\widehat{\rho}(Tu_h)_{|e} = \begin{cases} \{Tu_h\} & \text{if } e \in \mathcal{E}_0 \cup \mathcal{E}_N \\ \chi_h & \text{if } e \in \mathcal{E}_D \end{cases} \tag{4.6}$$

$$\widehat{\sigma}(Tu_h)_{|e} = \begin{cases} \{a_h\nabla_h Tu_h\} - \mu \, [\![\, \Pi_h Tu_h \,]\!] & \text{if } e \in \mathcal{E}_0 \\ a_h \nabla_h Tu_h - \mu(\Pi_h Tu_h - \chi_h)n & \text{if } e \in \mathcal{E}_D \\ 0 & \text{if } e \in \mathcal{E}_N \end{cases} , \tag{4.7}$$

and for the LDG scheme, the numerical fluxes read

$$\widehat{\rho}(Tu_h)_{|e} = \begin{cases} \{Tu_h\} - \theta \cdot [\![\, Tu_h \,]\!] & \text{if } e \in \mathcal{E}_0 \\ \chi_h & \text{if } e \in \mathcal{E}_D \\ Tu_h & \text{if } e \in \mathcal{E}_N \end{cases} \tag{4.8}$$

$$\widehat{\sigma}(Tu_h, \sigma_h)_{|e} = \begin{cases} \{\sigma_h\} + \theta \cdot [\![\, \sigma_h \,]\!] - \mu \, [\![\, \Pi_h Tu_h \,]\!] & \text{if } e \in \mathcal{E}_0 \\ \sigma_h - \mu(\Pi_h Tu_h - \chi_h)n & \text{if } e \in \mathcal{E}_D \\ 0 & \text{if } e \in \mathcal{E}_N \end{cases} , \tag{4.9}$$

with μ defined by (3.7). We notice that, as for the symmetric case, the numerical fluxes for the exponentially fitted schemes are single valued.

To complete the definition of the numerical schemes (4.5) we still have to specify the piecewise constant approximation a_h of a. We define

$$a_{h|K} := \mathcal{H}_K(a) =: \frac{1}{\fint_K \, a^{-1} \, dx}. \tag{4.10}$$

We remark that the presence of exponentials in a (we recall that here $a = \varepsilon e^{\psi/\varepsilon}$) and in the transformation $u_h \to \rho_h$ is a potential source of numerical troubles. However, thanks to the structure of the DG schemes developed in section 3, as well as to the choice of a_h as in (4.10) and of the operator T as in (4.2), system (4.5) scales well in terms of the exponentials and no overflow occurs in the implementation. Indeed, in the first equation of (4.5), for a given K, we can take $\tau \in \Sigma_h$ with $\tau \neq (0,0)$ in K and $\tau \equiv (0,0)$ elsewhere, and rescale the equation by multiplication with $e^{\psi_{m,K}/\varepsilon}$ (here and in the following we use the notation of section 2 for the minimum of ψ on an element K or on an edge e of K). Thus, it is easy to see that only exponentials with negative exponents enter the formula. This is obvious for the first term and, due to the trivial observation that $\psi_{m,K} \leq \psi_{m,e,K}$, also for the second term. This inequality is useful also for the third term. Indeed, since τ is piecewise constant, only the value in the midpoint of edge e of $\widehat{\rho}(Tu_h)$ enters the integrals in the third term. Therefore, since ψ/ε is continuous across e, we have

$$\int_{\partial K} \{Tu_h\}\tau \cdot n_K \, ds = \sum_{i=1}^{3} E(K, e^i) \int_{e_i} \{u_h\}\tau \cdot n_K \, ds$$

and

$$\int_{\partial K} [\![\, Tu_h \,]\!] \, \tau \cdot n_K \, ds = \sum_{i=1}^{3} E(K, e^i) \int_{e_i} [\![\, u_h \,]\!] \, \tau \cdot n_K \, ds,$$

so that we can conclude. We point out that the use of the transformation (4.3) in the discretization of the symmetric problem with standard IP or standard LDG schemes would produce an overflow at this point, since in that cases the vector variable is a polynomial of degree 1 on each element K.

In the second equation of (4.5), with $\widehat{\sigma}(Tu_h)$ defined in (4.7), the term $\{a_h \nabla_h T u_h\}_{|e}$ contains coefficients of the form $E(K_i, e)\mathcal{H}_{K_i}(a)$, for the two elements K_1 and K_2 sharing the edge e. Using again $\psi_{m,K_i} \leq \psi_{m,e,K_i}$, for $i = 1, 2$, that product can be rewritten with exponentials with negative exponents and no overflow occurs. In considering the term $\mu [\![\Pi_h T u_h]\!]$, which appears both in (4.7) and in (4.9), the presence of the projection Π_h and the continuity of ψ/ε across the edge e are crucial. The penalty coefficient $\mu_{|e}$ defined in (3.7) contains the diffusion coefficient a_h both on K_1 and K_2. However, since $E(K_1, e) = E(K_2, e)$, the jump term can be simplified to

$$[\![\Pi_h T u_h]\!]_e = E(K, e) [\![\Pi_h u_h]\!]_e, \tag{4.11}$$

where K could be either one from K_1 or K_2. As above, we have terms of the form $E(K_i, e)\mathcal{H}_{K_i}(a)$, which behave well, since we compare the minimum of the function ψ_{K_i} on the element K_i with the minimum of ψ_{K_i} on an edge of K_i. We shall see in the second part of the section how we can modify the definition of the numerical fluxes in order to extend the scheme to the case ψ/ε discontinuous across the interelement boundaries.

Remark 4.1. We point out that using the mean value of a instead of the harmonic average in definition (4.10) may produce overflow in the implementation of the scheme. Indeed, we would have to rescale the first equation of (4.5) by multiplication with $e^{\psi_{M,K}/\varepsilon}$, and, in that case, the second term would take the form $E(K, e)e^{-\psi_{M,K}/\varepsilon}$, which scales as $e^{(\psi_{M,K}-\psi_{m,e,K})/\varepsilon}$, where the exponent is positive (and possibly large).

Elimination of the vector variable σ_h as performed in section 3 leads to the primal formulation of the schemes (4.5). The primal formulation of the exponentially fitted IP scheme takes the form: find $u_h \in V_h$, such that, for all $v \in V_h$

$$C_h^{IP}(u_h, v) = \int_\Omega f v \, dx - \int_{\Gamma_D} \left(\chi_h n \cdot \{a_h \nabla_h v\} - \mu \chi_h n \cdot [\![v]\!] \right) \, ds \quad , \tag{4.12}$$

with the bilinear form $C_h^{IP} : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ given by

$$\begin{aligned} C_h^{IP}(u_h, v) = & \int_\Omega a_h \nabla_h T u_h \cdot \nabla_h v \, dx - \int_{\Gamma_0 \cup \Gamma_D} [\![T u_h]\!] \cdot \{a_h \nabla_h v\} \, ds \\ & + \int_{\Gamma_0 \cup \Gamma_D} [\![v]\!] \cdot \{a_h \nabla_h T u_h\} \, ds + \int_{\Gamma_0 \cup \Gamma_D} \mu [\![\Pi_h T u_h]\!] \cdot [\![\Pi_h v]\!] \, ds. \end{aligned} \tag{4.13}$$

The primal formulation for the exponentially fitted LDG scheme is obtained similarly and we do not report its explicit form.

Remark 4.2. We point out that the same primal formulation is obtained starting from the primal formulation (3.11)-(3.12) of the symmetric problem, taking a_h as in (4.10) and substituting ρ_h with Tu_h. In other words, recalling that the degrees of freedom in V_h are the midpoint of the edges, the algebraic form of (4.12)-(4.13) is obtained performing a change of variable in the algebraic form of the symmetric problem with a right multiplication by a diagonal matrix with $E(K, e)$ as entries (and a proper modification of the right hand side in the part corresponding to the Dirichlet boundary data). The same comment applies to the LDG case. Moreover, since $E(K, e)\mathcal{H}_K(a)$ vanishes asymptotically (in ε) when $\psi_{m,e,K} > \psi_{m,K}$, the corresponding entry in the matrix vanishes, and an automatic upwind effect is produced (see [19] for details in the hybrid and mixed finite element framework).

Remark 4.3. As pointed out in Lemma 3.1, the penalty coefficients μ must be large in order to guarantee the coercivity property, or for LDG it must be at least bounded away from zero. However, the presence of the exponentials could produce degeneracy at Dirichlet boundary nodes. This is not the case at the inflow Dirichlet boundary, but it could happen at the outflow boundary. Therefore, we modify the definition of μ as follows. If the edge e is at the outflow boundary and it is contained in the element K, taking into account definition (3.7), we choose $\mu_{|e}$ such that

$$\mu_{|e} E(K,e) = \frac{\eta_e}{h_e} \max\{1, E(K,e)\mathcal{H}_K(a)\}.$$

With this choice, the numerical flux becomes, for instance, for IP

$$\widehat{\sigma} = a_h \nabla_h T u_h - \frac{\eta_e}{h_e} \max\{1, E(K,e)\mathcal{H}_K(a)\} (\Pi_h u_h - g_h) n.$$

4.2. **The general case.** We conclude the section introducing the exponentially fitted schemes for a general class of convection–diffusion equations. Here, we allow $\varepsilon(x)$ and $\beta(x)$ to be piecewise constant functions. A *global* transformation of the form (1.3) does not exist in the present case. However, there exists a piecewise linear ψ whose gradient coincides with the constant β_K on the element K (i.e., $\beta_K = \nabla \psi_K$). Then, we define transformation $u \longrightarrow \rho$ only *locally* as

$$\rho_K = u_K e^{-\frac{\psi}{\varepsilon}|_K}, \quad \text{in } K. \tag{4.14}$$

One can define again the operator $T : V_h \longrightarrow V_h$ as in (4.2), but in this case the function ψ/ε is discontinuous across the interelement boundaries, and the schemes (4.5) may exhibit overflows in the third term of the first equation and in the penalization term in the second equation, as already pointed out in section 4.1.

In order to be able to deal with the generic case of ψ/ε piecewise linear, we change the definition of the numerical fluxes. We replace definitions (4.6) and (4.7) for the IP scheme with

$$\widehat{u}_K(u_h)_{|e} = \begin{cases} E(K,e)\{u_h\} & \text{if } e \in \mathcal{E}_0 \cup \mathcal{E}_N \\ \chi_h & \text{if } e \in \mathcal{E}_D \end{cases} \tag{4.15}$$

$$\widehat{\sigma}(u_h)_{|e} = \begin{cases} \{a_h \nabla_h T u_h\} - \widetilde{\mu} [\![\Pi_h u_h]\!] & \text{if } e \in \mathcal{E}_0 \\ a_h \nabla_h T u_h - \widetilde{\mu}(\Pi_h u_h - g_h) n & \text{if } e \in \mathcal{E}_D \\ 0 & \text{if } e \in \mathcal{E}_N, \end{cases} \tag{4.16}$$

and we replace definitions (4.8) and (4.9) for the LDG scheme with

$$\widehat{u}_K(u_h)_{|e} = \begin{cases} E(K,e)(\{u_h\} - \theta \cdot [\![u_h]\!]) & \text{if } e \in \mathcal{E}_0 \\ \chi_h & \text{if } e \in \mathcal{E}_D \\ E(K,e) u_h & \text{if } e \in \mathcal{E}_N \end{cases} \tag{4.17}$$

$$\widehat{\sigma}(u_h, \sigma_h)_{|e} = \begin{cases} \{\sigma_h\} + \theta \cdot [\![\sigma_h]\!] - \widetilde{\mu} [\![\Pi_h u_h]\!] & \text{if } e \in \mathcal{E}_0 \\ \sigma_h - \widetilde{\mu}(\Pi_h u_h - g_h) n & \text{if } e \in \mathcal{E}_D \\ 0 & \text{if } e \in \mathcal{E}_N \end{cases}. \tag{4.18}$$

The new penalization parameter $\widetilde{\mu}$ is taken as

$$\widetilde{\mu}|_e = \frac{\widetilde{\eta}_e}{h_e}, \quad \text{with} \quad \widetilde{\eta}_e = \frac{1}{2}\left(E(K,e) a_h|_K + E(K',e) a_h|_{K'}\right), \tag{4.19}$$

for $e = K \cap K'$, and with the obvious changes for boundary edges (see Remark 4.3).

Notice that, for both schemes, the definition of $\widehat{u}_K(u_h)$ is linked to the definition of numerical flux for the scalar variable in the symmetric problem through

$$\widehat{u}_K(u_h)_{|e} = E(K,e)\widehat{\rho}(u_h). \tag{4.20}$$

Moreover, the lack of continuity of ψ/ε across the interelement boundaries implies that the numerical flux $\widehat{u}$ is double valued on the internal edges.

The exponentially fitted Discontinuous Galerkin schemes can be written in the mixed compact form as follows: Find $\sigma_h \in \Sigma_h$ and $u_h \in V_h$ such that for all $\tau \in \Sigma_h$ and $v \in V_h$ it holds

$$\begin{aligned}\int_\Omega a_h^{-1}\sigma_h \cdot \tau\, dx + \int_\Omega Tu_h \mathrm{div}_h\,\tau\, dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{u}_K\tau\cdot n_K\, ds &= 0\\ \int_\Omega \sigma_h\cdot\nabla_h v\, dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{\sigma}\cdot n_K v\, ds &= \int_\Omega fv\, dx\,.\end{aligned} \tag{4.21}$$

One can easily check that overflow does not occur when the new definitions of the numerical fluxes are used.

Remark 4.4. We point out that when ψ/ε is continuous, the scheme (4.21) with the fluxes defined by (4.15) and (4.16) ((4.17) and (4.18), resp.) coincides with the scheme (4.5), with the fluxes given by (4.6) and (4.7) ((4.8) and (4.9), resp.). Although $\widehat{u}_K(u_h)_{|e}$ is different from $\widehat{\rho}(Tu_h)_{|e}$, even for a continuous ψ/ε, the integrals on the edges of the scalar fluxes against constant functions coincide (and $\tau\cdot n\in\mathcal{P}_0(e)$).

In order to eliminate the variable σ_h for writing the primal formulation, we follow the same procedure as in section 3 (see (3.8)), but here the definition of the lifting operators needs some extra care.

Let us introduce the function $\widetilde{E}\in Tr(\Gamma)$, which is double valued on each interior edge and single valued on $\partial\Omega$, as follows. Let e be an interior edge shared by two elements K_1 and K_2. Then, on e, we define $\widetilde{E}$ taking the values $\widetilde{E}_i = E(K_i, e)$, $i = 1, 2$. We introduce the lifting operators $\widetilde{r} : [L^2(\Gamma)]^2 \to \Sigma_h$ and $\widetilde{l} : L^2(\Gamma_0) \to \Sigma_h$ defined by

$$\begin{aligned}\int_\Omega \widetilde{r}(\phi)\cdot\tau\, dx &= -\int_\Gamma \phi\cdot\{\widetilde{E}a_h\tau\}\, ds, \quad \forall\tau\in\Sigma_h,\\ \int_\Omega \widetilde{l}(q)\cdot\tau\, dx &= -\int_{\Gamma_0} q\,[\![\,\widetilde{E}a_h\tau\,]\!]\, ds, \quad \forall\tau\in\Sigma_h.\end{aligned} \tag{4.22}$$

Similar arguments like the ones discussed in section 4.1, show that the exponentials in $\widetilde{E}a_h$ are well balanced.

When considering the mid–term in the first equation of (4.21), we can integrate by parts on each triangle to obtain

$$\begin{aligned}\int_\Omega Tu_h\mathrm{div}_h\,\tau\, dx &= -\int_\Omega \nabla_h Tu_h\cdot\tau\, dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K} Tu_h\,\tau\cdot n_K\, ds\\ &= -\int_\Omega \nabla_h Tu_h\cdot\tau\, dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K} u_h\,\widetilde{E}\tau\cdot n_K\, ds\\ &= -\int_\Omega \nabla_h Tu_h\cdot\tau\, dx + \int_\Gamma [\![\,u_h\,]\!]\cdot\{\widetilde{E}\tau\}\, ds + \int_{\Gamma_0}\{u_h\}\,[\![\,\widetilde{E}\tau\,]\!]\, ds,\end{aligned}$$

where, from the first to the second line, we used $\tau\cdot n\in\mathcal{P}_0(e)$ and the equality $(\Pi_h Tu_h)_K|_e = E(K,e)\Pi_h u_h|_e$. Similarly, the third term in the first equation of (4.21) becomes

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{u}_K\tau\cdot n_K\, ds = \int_\Gamma [\![\,\widehat{\rho}(u_h)\,]\!]\cdot\{\widetilde{E}\tau\}\, ds + \int_{\Gamma_0}\{\widehat{\rho}(u_h)\}\,[\![\,\widetilde{E}\tau\,]\!]\, ds,$$

where we used (4.20). Therefore, from the first equation of (4.21) we have that for all $\tau \in \Sigma_h$

$$\int_\Omega a_h^{-1}\sigma_h \cdot \tau \, dx - \int_\Omega \nabla_h T u_h \cdot \tau \, dx + \int_\Gamma [\![\, u_h - \widehat{\rho}(u_h) \,]\!] \cdot \{\widetilde{E}\tau\} \, ds + \int_{\Gamma_0} \{u_h - \widehat{\rho}(u_h)\} [\![\, \widetilde{E}\tau \,]\!] \, ds = 0. \tag{4.23}$$

Taking $\tau = a_h\varphi$ for all $\varphi \in \Sigma_h$ in (4.23), we can eliminate σ_h obtaining

$$\sigma_h = a_h \nabla_h T u_h - \widetilde{r}(\, [\![\, \widehat{\rho}(u_h) - u_h \,]\!] \,) - \widetilde{l}(\{\widehat{\rho}(u_h) - u_h\}), \tag{4.24}$$

where the liftings $\widetilde{r}$ and $\widetilde{l}$ are defined in (4.22).

Plugging (4.24) into the second equation of (4.21) we obtain the primal formulation of the methods: Find $u_h \in V_h$ such that, for all $v \in V_h$, it holds

$$\int_\Omega a_h \nabla_h T u_h \cdot \nabla_h v \, dx + \int_\Gamma (\, [\![\, \widehat{\rho}(u_h) - u_h \,]\!] \cdot \{\widetilde{E} a_h \nabla_h v\} - \{\widehat{\sigma}\} \cdot [\![\, v \,]\!] \,) \, ds + \int_{\Gamma_0} (\{\widehat{\rho}(u_h) - u_h\} [\![\, \widetilde{E} a_h \nabla_h v \,]\!] - [\![\, \widehat{\sigma} \,]\!] \cdot \{v\}) \, ds = \int_\Omega f v \, dx. \tag{4.25}$$

Using the definition (4.15)-(4.16) of the numerical fluxes as done in section 3, in the case of the IP scheme equation (4.25) becomes

$$\int_\Omega a_h \nabla_h T u_h \cdot \nabla_h v \, dx - \int_{\Gamma_0 \cup \Gamma_D} [\![\, v \,]\!] \cdot (\{a_h \nabla_h T u_h\} - \widetilde{\mu} [\![\, \Pi_h u_h \,]\!] \,) \, ds - \int_{\Gamma_0 \cup \Gamma_D} [\![\, u_h \,]\!] \cdot \{\widetilde{E} a_h \nabla_h v\} \, ds + \int_{\Gamma_D} \Big(g_h n \cdot \{\widetilde{E} a_h \nabla_h v\} - [\![\, v \,]\!] \cdot \widetilde{\mu} g_h n \Big) \, ds = \int_\Omega f v \, dx. \tag{4.26}$$

Therefore, (4.25) takes the form

$$\widetilde{C}_h(u_h, v) = \int_\Omega f v \, dx - \int_{\Gamma_D} \Big(g_h n \cdot \{\widetilde{E} a_h \nabla_h v\} - \widetilde{\mu} g_h v \Big) \, ds \qquad \forall v \in V_h, \tag{4.27}$$

with the bilinear form $\widetilde{C}_h = \widetilde{C}_h^{IP} : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ given by

$$\begin{aligned} \widetilde{C}_h^{IP}(u_h, v) := & \int_\Omega a_h \nabla_h T u_h \cdot \nabla_h v \, dx \\ & - \int_{\Gamma_0 \cup \Gamma_D} \Big([\![\, u_h \,]\!] \cdot \{\widetilde{E} a_h \nabla_h v\} + [\![\, v \,]\!] \cdot \{a_h \nabla_h T u_h\} \Big) \, ds \\ & + \int_{\Gamma_0 \cup \Gamma_D} \widetilde{\mu} [\![\, \Pi_h u_h \,]\!] \cdot [\![\, \Pi_h v \,]\!] \, ds. \end{aligned} \tag{4.28}$$

Taking into account the new definition of the liftings (4.22), in the case of the LDG scheme, (4.25) takes the form (4.27), with $\widetilde{C}_h = \widetilde{C}_h^{LDG} : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ defined by

$$
\begin{aligned}
\widetilde{C}_h^{LDG}(u_h, v) := & \int_\Omega a_h \nabla_h T u_h \cdot \nabla_h v dx \\
& - \int_{\Gamma_0 \cup \Gamma_D} \Big([\![u_h]\!] \cdot \{\widetilde{E} a_h \nabla_h v\} + \{a_h \nabla_h T u_h\} \cdot [\![v]\!] \Big) ds \\
& - \int_{\Gamma_0} \Big([\![a_h \nabla_h T u_h]\!] \theta \cdot [\![v]\!] + [\![u_h]\!] \cdot \theta [\![\widetilde{E} a_h \nabla_h v]\!] \Big) ds \\
& + \int_{\Gamma_0 \cup \Gamma_D} \widetilde{\mu} [\![\Pi_h u_h]\!] \cdot [\![\Pi_h v]\!] ds \\
& + \int_{\Gamma_0 \cup \Gamma_D} \{\widetilde{\Upsilon}\} \cdot [\![v]\!] ds + \int_{\Gamma_0} \theta [\![\widetilde{\Upsilon}]\!] \cdot [\![v]\!] ds,
\end{aligned} \tag{4.29}
$$

where

$$\widetilde{\Upsilon} = -\widetilde{r}([\![u_h]\!]) - \widetilde{l}(\theta \cdot [\![u_h]\!]).$$

Remark 4.5. We point out that, once $\widetilde{\mu}$ is chosen as in (4.19), in principle, the projection Π_h is not needed in the primal formulation for controlling the exponentials in that term. However, numerical experiments show that the corresponding scheme exhibits a stronger artificial diffusion (see Remark 5.1).

5. Numerical examples

We present some numerical tests of the methods proposed in the previous section in the IP case. The examples below refer to a constant diffusion coefficient with the value $\varepsilon = 10^{-6}$. The transport β is chosen irrotational in the first three tests and scheme (4.12)-(4.13) has been used in the code. The last experiment has been designed to assess the performance of the scheme (4.27)-(4.28).

Test 1. In this first test, we deal with a boundary layer case. We present an example for which we know the exact solution, in order to study the convergence of the scheme numerically. Equation (1.1) is considered in the domain $\Omega = (-1,1)^2$ with $\beta = [1,1]$, with homogeneous Dirichlet boundary condition on $\partial\Omega$ and with the right hand side f given by

$$f(x,y) = \frac{1 + e^{-\frac{2}{\varepsilon}} - 2e^{\frac{x-1}{\varepsilon}}}{1 - e^{-\frac{2}{\varepsilon}}} + x + \frac{1 + e^{-\frac{2}{\varepsilon}} - 2e^{\frac{y-1}{\varepsilon}}}{1 - e^{-\frac{2}{\varepsilon}}} + y.$$

The exact solution is then

$$u(x,y) = \left(\frac{1 + e^{-\frac{2}{\varepsilon}} - 2e^{\frac{x-1}{\varepsilon}}}{1 - e^{-\frac{2}{\varepsilon}}} + x \right) \left(\frac{1 + e^{-\frac{2}{\varepsilon}} - 2e^{\frac{y-1}{\varepsilon}}}{1 - e^{-\frac{2}{\varepsilon}}} + y \right).$$

The numerical solution, shown in Figure 3, exhibits sharp boundary layers, with no wiggles and virtually without numerical diffusion. We point out that the boundary conditions are well treated even at the outflow boundary, thanks to the choice of μ as in Remark 4.3.

In the Table 1 we study the numerical order of convergence in norm L^2 for both variables u and σ. The orders of convergence obtained are 0.49669 and 0.48668 respectively. An optimal order of convergence (with respect to h) is not to be expected here, because of the singularly perturbed behavior of the exact solution.

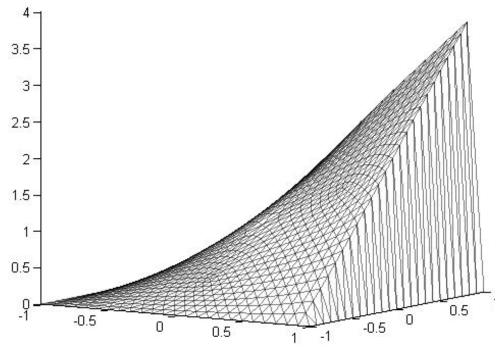

FIGURE 3. Test 1.

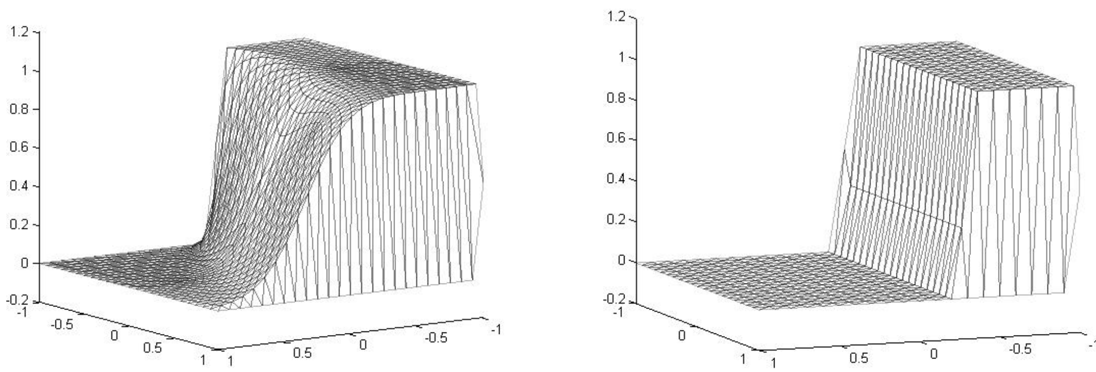

FIGURE 4. Left: Test 2 with $\beta = [2/3, 2]$ and an unstructured mesh; Right: Test 2 with $\beta = [0, 1]$ and a mesh aligned to the transport direction.

Test 2. We study here a propagation of a boundary discontinuity in the case of a constant transport vector. In the domain $\Omega = (-1, 1)^2$ we take $\beta = [\frac{2}{3}, 2]$, $f = 0$ and $\Gamma_D = \partial\Omega$. The boundary condition is

$$g(x, y) = \begin{cases} 1 \text{ if } x = -1 \text{ or } x < -\frac{1}{3} \text{ and } y = -1 \\ 0 \text{ elsewhere.} \end{cases}$$

The solution is displayed in Figure 4 (left). The proposed scheme suffers of a quite strong crosswind diffusion. This unpleasant behavior, however, does not appear when the mesh is aligned to the transport direction. Figure 4 (right) shows the solution for $\beta = [0, 1]$ and

$$g(x, y) = \begin{cases} 1 \text{ if } x = -1 \text{ or } x < -\frac{1}{3} \text{ and } y = -1 \\ 0 \text{ elsewhere.} \end{cases}$$

Test 3. This test is taken from [16]. The problem is partly advection dominated, and partly diffusion dominated. Moreover, the right hand side f varies abruptly in the advection dominated portion of the domain. We solve equation (1.1) in the domain $\Omega = (-1, 1)^2$

with $\beta = \nabla\psi$, where

$$\psi = \psi(r) = \begin{cases} 0 & \text{if } r < 1.5 \\ 2(r-1.5) & \text{if } 1.5 \le r < 1.9, \\ 0.8 & \text{if } 1.9 \le r \end{cases}$$

with $r^2 = (x+1)^2 + (y+1)^2$, and f given by

$$f = f(r) = \begin{cases} 0 & \text{if } r < 1.6 \\ \frac{1}{2}\varepsilon 10^{16} & \text{if } 1.6 \le r < 1.8. \\ 0 & \text{if } 1.8 \le r \end{cases}$$

The Dirichlet boundary is $\Gamma_D = [0.6, 1] \times \{y = 1\} \cup \{x = 1\} \times [0.6, 1] \cup [-1, -0.6] \times \{y = -1\} \cup \{x = -1\} \times [-1, -0.6]$ and we consider homogeneous Neumann condition on the rest of the boundary $\Gamma_N = \partial\Omega - \Gamma_D$. The Dirichlet condition is given by

$$g(x,y) = \begin{cases} 10^{17} & \text{if } (x,y) \in [0.6,1] \times \{y=1\} \cup \{x=1\} \times [0.6,1] \\ 10^3 & \text{if } (x,y) \in [-1,-0.6] \times \{y=-1\} \cup \{x=-1\} \times [-1,-0.6]. \end{cases}$$

We point out that the mesh used for this example is not adapted to the solution. We can see in Figure 5 that no oscillation occurs and that the internal layers are well reproduced, without appearance of numerical diffusion. We remark also that the undershooting along the arc $r = 1.5$ is present in the actual solution and is not due to numerical pollution.

Test 4. Now we explore the case of a rotating flow transporting a boundary sharp profile. In the domain $\Omega = (-1,1) \times (0,1)$ we consider the equation (1.1) with

$$\beta(x,y) = \left(2y(1-x^2), -2x(1-y^2)\right), \qquad \text{and} \qquad f = 0.$$

We also have $\Gamma_D = \partial\Omega$ and the Dirichlet condition is

$$g(x,y) = \begin{cases} 1 + \tanh(10(2x+1)) & \text{if } x \le 0 \text{ and } y = 0 \\ 0 & \text{elsewhere.} \end{cases}$$

In this case we have to apply scheme (4.27)-(4.28) . For each triangle K in the triangulation we approximate $\beta|_K \sim \nabla\psi|_K$, where

$$\psi|_K = 2y_K(1 - x_K^2)\,x - 2x_K(1 - y_K^2)\,y,$$

where (x_K, y_K) is the barycenter of K.

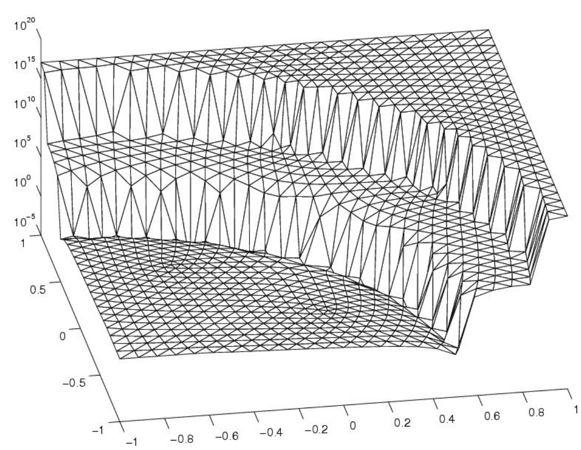

Figure 5. Test 3, in semilogarithmic scale.

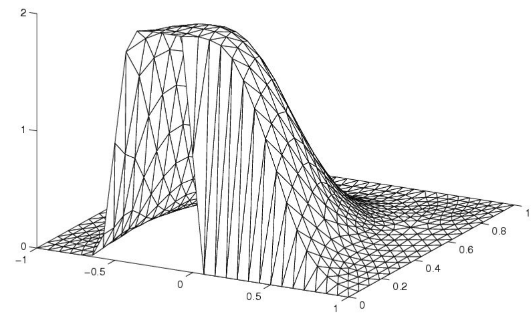

FIGURE 6. Test 4.

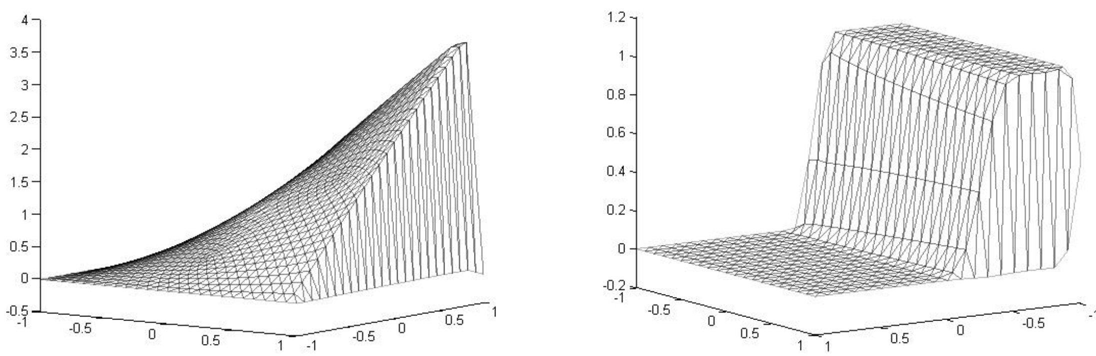

FIGURE 7. Tests described in Remark 5.1.

We display the obtained solution in Figure 6. As in Test 2, crosswind diffusion shows up in this case. We can then say that this behavior is intrinsic to the exponential fitting procedure, and not to the piecewise constant approximation of the transport vector (see also the numerical test presented in [18, 19]).

Remark 5.1. At the end of section 4 we pointed out that with choice of μ as in (4.19) the projection could be removed. However, in this case, the larger connectivity (the same as for the standard IP scheme for the symmetric case) has the effect to introduce extra numerical diffusion. This is apparent comparing Figure 3 with Figure 7 (left), which presents the solution for Test 1, when the projection is dropped. It is even more evident, comparing Figure 4 (right) with Figure 7 (right).

6. Error estimates for the symmetric problem

We collect in this section the proofs of Theorems 3.2, 3.4 and 3.5. For shortening the presentation, we consider the simplest case $a = 1$, $\Gamma_D = \partial\Omega$, and $\chi = 0$ (corresponding to Laplace equation with homogeneous Dirichlet boundary conditions). The convexity of the domain Ω implies the elliptic regularity of the solution ρ.

The proof of the error estimates for $|||\rho - \rho_h|||_h$ is based on the primal formulation (3.10) of the discrete problem, and it follows from the standard ingredients of consistency, stability and boundedness of the discrete bilinear form. For reader's convenience, we report here the discrete primal formulation in the simplified setting. Find $\rho_h \in V_h$ such that

$$B_h(\rho_h, v) = \int_\Omega f v \, dx \qquad \forall v \in V_h,$$

where $B_h : H^2(\mathcal{T}_h)^2 \to \mathbb{R}$ is given by

$$B_h(\rho_h, v) = \int_\Omega \nabla_h \rho_h \cdot \nabla_h v \, dx - \int_\Gamma ([\![\rho_h]\!] \cdot \{\nabla_h v\} + \{\nabla_h \rho_h\} \cdot [\![v]\!]) \, ds + \int_\Gamma \mu [\![\Pi_h \rho_h]\!] \cdot [\![\Pi_h v]\!] + \Xi(\rho_h, v) \tag{6.1}$$

where for the IP method we have $\Xi = 0$, while for LDG we have

$$\Xi(\rho_h, v) = -\int_{\Gamma_0} ([\![\nabla_h \rho_h]\!] \theta \cdot [\![v]\!] + [\![\rho_h]\!] \cdot \theta [\![\nabla_h v]\!]) \, ds + \int_\Omega (r([\![\rho_h]\!]) + l(\theta \cdot [\![\rho_h]\!])) \cdot (r([\![v]\!]) + l(\theta \cdot [\![v]\!])) \, dx.$$

Consistency of the scheme can be easily checked by using integration by parts and the $H^2(\Omega)$-regularity of the exact solution ρ to get

$$B_h(\rho, v) = \int_\Omega f v \, dx \qquad \forall v \in H^2(\mathcal{T}_h). \tag{6.2}$$

On the space $H^2(\mathcal{T}_h)$ we consider the norm $||| \cdot |||_h$ defined in section 2.

In order to prove boundedness of the bilinear form $B_h(\cdot, \cdot)$ the following standard trace inequality will be useful: for all $K \in \mathcal{T}_h$ and e edge of K it holds

$$\|q\|_{0,e}^2 \le C \left(h_e^{-1} \|q\|_{0,K}^2 + h_e |q|_{1,K}^2 \right) \qquad \forall q \in H^1(K), \tag{6.3}$$

where the constant C depends only on the minimum angle of $\mathcal{T}_h$. Also we have the Poincaré type inequality

$$\|q - \Pi_0^e q\|_{0,K} \le C h_K |q|_{1,K} \qquad \forall q \in H^1(K). \tag{6.4}$$

From these two inequalities we obtain

$$\| [\![q - \Pi_h q]\!] \|_{0,e}^2 \le C h_e \sum_{K \in \mathcal{T}_h : e \subseteq \partial K} |q|_{1,K}^2. \tag{6.5}$$

We also need the following Lemma. Here, the function $h_\mathcal{T}$ is defined as $h_\mathcal{T}(x) = h_K$, for x in the interior of the element $K \in \mathcal{T}_h$. In what follows, θ is a piecewise constant function on Γ, as in the definition of the fluxes for the modified LDG scheme.

Lemma 6.1. *We have, for all $q \in Tr(\Gamma)$,*

$$\|r([\![q]\!])\|_{0,\Omega} \le C|q|_*, \qquad \|l(\theta \cdot [\![q]\!])\|_{0,\Omega} \le C|q|_*$$
$$\left\| h_\mathcal{T}^{\frac{1}{2}} r([\![q]\!]) \right\|_{0,\Omega} \le C|q|_\star, \qquad \left\| h_\mathcal{T}^{\frac{1}{2}} l(\theta \cdot [\![q]\!]) \right\|_{0,\Omega} \le C|q|_\star.$$

where C depends only on the minimum angle of the mesh, on the function η (appearing in the definition (3.20) of the seminorm $|\cdot|_\star$) and on θ.

Proof. We prove the third inequality, the others follow analogously. Fix the element K. We have

$$\|r(\llbracket \phi \rrbracket)\|_{0,K}^2 = \int_K r(\llbracket \phi \rrbracket) \cdot r(\llbracket \phi \rrbracket)\, dx = \int_\Omega r(\llbracket \phi \rrbracket) \cdot \tau\, dx$$

where τ is the function verifying $\tau(x) = r(\llbracket \phi \rrbracket)$ if $x \in K$ and vanishing outside K. Notice that $\tau \in \Sigma_h = \mathcal{P}_0(\mathcal{T}_h)^2$. Then, from the definition (3.9) of the lifting r, it follows

$$\begin{aligned}\|r(\llbracket \phi \rrbracket)\|_{0,K}^2 &= -\int_\Gamma \llbracket \phi \rrbracket \cdot \{\tau\}\, ds = -\int_\Gamma \llbracket \Pi_h \phi \rrbracket \cdot \{\tau\}\, ds \\ &\leq \sum_{e\in\mathcal{E}: e\subseteq\partial K} h_e^{-\frac{1}{2}} \| \llbracket \Pi_h \phi \rrbracket \|_{0,e} \|r(\llbracket \phi \rrbracket)\|_{0,K} \\ &\leq C h_K^{-\frac{1}{2}} \|r(\llbracket \phi \rrbracket)\|_{0,K} \sum_{e\in\mathcal{E}: e\subseteq\partial K} \| \llbracket \Pi_h \phi \rrbracket \|_{0,e}.\end{aligned}$$

So, $h_K^{\frac{1}{2}} \|r(\llbracket \phi \rrbracket)\|_{0,K} \leq C \sum_{e\in\mathcal{E}: e\subseteq\partial K} \| \llbracket \Pi_h \phi\|_{0,e}$. Therefore,

$$\begin{aligned}\left\| h_{\mathcal{T}}^{\frac{1}{2}} r(\llbracket \phi \rrbracket) \right\|_{0,\Omega}^2 &= \sum_{K\in\mathcal{T}_h} h_K \|r(\llbracket \phi \rrbracket)\|_{0,K}^2 \\ &\leq C \sum_{K\in\mathcal{T}_h} \left(\sum_{e\in\mathcal{E}: e\subseteq\partial K} \| \llbracket \Pi_h \phi \rrbracket \|_{0,e} \right)^2 \leq C|\phi|_\star^2,\end{aligned}$$

as we wanted. ☐

Proposition 6.2. *Let $B_h(.,.)$ be the bilinear form introduced in (6.1) either for the IP or LDG schemes, with the penalization parameter μ given by (3.7). Then, we have*

$$|B_h(w,v)| \leq C_b |||w|||_h \, |||v|||_h \qquad \forall w, v \in H^2(\mathcal{T}_h)^2. \tag{6.6}$$

with C_b a constant depending only on the regularity of the mesh, on the function η and, for the LDG scheme, on the parameter θ .

Proof. For $w, v \in H^2(\mathcal{T}_h)$ we have

$$B_h(w,v) = \int_\Omega \nabla_h w \cdot \nabla_h v\, dx - \int_\Gamma (\llbracket w \rrbracket \cdot \{\nabla_h v\} + \llbracket v \rrbracket \cdot \{\nabla_h w\})\, ds + A_1 + A_2,$$

where, for the IP method,

$$A_1 = 0, \qquad A_2 = \int_\Gamma \mu \llbracket \Pi_h w \rrbracket \cdot \llbracket \Pi_h v \rrbracket \, ds,$$

and for LDG we have

$$\begin{aligned}A_1 &= -\int_{\Gamma_0} (\llbracket \nabla_h w \rrbracket \theta \cdot \llbracket v \rrbracket + \llbracket \nabla_h v \rrbracket \theta \cdot \llbracket w \rrbracket)\, ds \\ A_2 &= \int_\Gamma \mu \llbracket \Pi_h w \rrbracket \cdot \llbracket \Pi_h v \rrbracket \, ds + \int_\Omega (r(\llbracket w \rrbracket) + l(\theta \cdot \llbracket w \rrbracket)) \cdot (r(\llbracket v \rrbracket) + l(\theta \cdot \llbracket v \rrbracket))\, dx.\end{aligned}$$

Clearly, using Lemma 6.1, we have easily

$$|A_2| \leq C|w|_*|v|_*.$$

Then, it is enough to estimate terms of the form

$$\int_\Gamma \llbracket v \rrbracket \cdot \{\nabla_h w\}\, ds$$

and of the form

$$\int_{\Gamma_0} [\![\nabla_h v]\!] \theta \cdot [\![w]\!] \, ds.$$

We shall estimate only the first one. The second term can be analyzed analogously.

Let e be an internal edge, with $e = K \cap K'$, and let $v \in H^2(\mathcal{T}_h)$. Then we have

$$\int_e \frac{\partial w_K}{\partial n} (v_K - v_{K'}) \, ds = \int_e \frac{\partial w_K}{\partial n} \Pi_0^e (v_K - v_{K'}) \, ds + \int_e \frac{\partial w_K}{\partial n} \left((v_K - v_{K'}) - \Pi_0^e (v_K - v_{K'}) \right) \, ds.$$

Since $\Pi_0^e(v_K - v_{K'}) = [\![\Pi_h v]\!] \cdot n^+$ and $v_K - v_{K'} = [\![v]\!] \cdot n^+$ it follows that

$$\left| \int_e \frac{\partial w_K}{\partial n} (v_K - v_{K'}) \, ds \right| \le h_e^{\frac{1}{2}} \left\| \frac{\partial w_K}{\partial n} \right\|_{0,e} \left(h_e^{-\frac{1}{2}} \| [\![\Pi_h v]\!] \|_{0,e} + h_e^{-\frac{1}{2}} \| [\![v - \Pi_h v]\!] \|_{0,e} \right).$$

From (6.5) we have

$$h_e^{-1} \| [\![v - \Pi_h v]\!] \|_{0,e}^2 \le C(|v|_{1,K}^2 + |v|_{1,K'}^2)$$

and, using the trace inequality (6.3), we have

$$\left| \int_e \frac{\partial w_K}{\partial n} (v_K - v_{K'}) \, ds \right| \le C \left(|w|_{1,K}^2 + h_K^2 |w|_{2,K}^2 \right)^{\frac{1}{2}} \left(h_e^{-1} \| [\![\Pi_h v]\!] \|_{0,e}^2 + |v|_{1,K}^2 + |v|_{1,K'}^2 \right)^{\frac{1}{2}}.$$

Similar estimates hold for the boundary edges, so that summation over all $e \in \mathcal{E}$ gives

$$\begin{aligned} \left| \int_\Gamma \{ \nabla_h w \} \cdot [\![v]\!] \, ds \right| &\le C \left(|w|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_T^2 |w|_{2,K}^2 \right)^{\frac{1}{2}} \left(|v|_{1,h}^2 + \sum_{e \in \mathcal{E}} h_e^{-1} \| [\![\Pi_h v]\!] \|_{0,e}^2 \right)^{\frac{1}{2}} \\ &\le C \, |||w|||_h \, |||v|||_h. \end{aligned}$$

Now (6.6) follows collecting the previous estimates. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. We proceed in the standard way [3]. Indeed, let $\rho_I \in V_h$ be the usual continuous interpolant of ρ which satisfies

$$|||\rho - \rho_I|||_h^2 = |\rho - \rho_I|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\rho - \rho_I|_{2,K}^2 \le C_a h^2 |\rho|_{2,\Omega}^2.$$

Then, using the stability (3.15), consistency (6.2), boundedness (6.6) and the above approximation property, we have

$$|||\rho_I - \rho_h|||_h \le \frac{C_b C_a}{C_s} h |\rho|_{2,\Omega}.$$

Thus, the first estimate of the statement is a consequence of the triangle inequality.

It remains to estimate the error for σ. Since $\sigma = \nabla \rho$ we have, using expression (3.8),

$$\sigma - \sigma_h = \nabla(\rho - \rho_h) + r([\![\widehat{\rho}(\rho_h) - \rho_h]\!]) + l(\{ \widehat{\rho}(\rho_h) - \rho_h \}).$$

Then, for IP it follows

$$\sigma - \sigma_h = \nabla(\rho - \rho_h) - r([\![\rho_h]\!])$$

while, for LDG we have

$$\sigma - \sigma_h = \nabla(\rho - \rho_h) - r([\![\rho_h]\!]) - l(\theta \cdot [\![\rho_h]\!]).$$

Now, from Lemma 6.1 we have

$$\|r(\llbracket \rho_h \rrbracket)\|_{0,\Omega} \leq C|\rho_h|_*$$
$$\|l(\theta \cdot \llbracket \rho_h \rrbracket)\|_{0,\Omega} \leq C|\rho_h|_*.$$

The proof ends by noting that $|\rho_h|_* = |\rho - \rho_h|_*$, and by using the estimate for $|||\rho - \rho_h|||_h$. □

In order to prove Theorem 3.4 we shall use the mixed formulation (3.17)-(3.19) introduced in section 3.

It is useful to rewrite the scheme (3.17)-(3.19) in a compact form as follows

$$\mathcal{A}(\sigma_h, \rho_h; \tau, v) = \int_\Omega f v \; dx, \qquad \forall (v, \tau) \in V_h \times \Sigma_h$$

with

$$\mathcal{A}(\sigma_h, \rho_h; \tau, v) = \int_\Omega \sigma_h \cdot \tau \; dx + b(\rho_h, \tau) - b(v, \sigma_h) + \int_\Gamma \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket \; ds.$$

From the consistency of the scheme, which can be easily checked, we have

$$\mathcal{A}(\sigma - \sigma_h, \rho - \rho_h; \tau, v) = 0 \qquad \forall (\tau, v) \in \Sigma_h \times V_h.$$

Proof of Theorem 3.4. Let σ_I be the L^2-projection of σ on Σ_h, and let ρ_I be the continuous piecewise linear interpolant of ρ. Then, we need to estimate $\|\sigma_h - \sigma_I\|_0 + |\rho_h - \rho_I|_\star$.

By the definition of $\mathcal{A}$ and the consistency of the scheme we have

$$\|\sigma_h - \sigma_I\|^2_{0,\Omega} + |\rho_h - \rho_I|^2_\star = \mathcal{A}(\sigma - \sigma_I, \rho - \rho_I; \sigma_h - \sigma_I, \rho_h - \rho_I). \tag{6.7}$$

To estimate the right hand side of this equality, we need to consider the following eight terms (which contribute to (6.7) after using the usual integration by parts in the div-term)

$$I_1 = \int_\Omega (\sigma - \sigma_I) \cdot (\sigma_h - \sigma_I)\, dx, \qquad I_2 = \int_\Omega \nabla(\rho - \rho_I) \cdot (\sigma_h - \sigma_I)\, dx,$$
$$I_3 = \int_\Gamma \llbracket \rho - \rho_I \rrbracket \cdot \{\sigma_h - \sigma_I\}\, ds, \qquad I_4 = \int_{\Gamma_0} \theta \cdot \llbracket \rho - \rho_I \rrbracket \llbracket \sigma_h - \sigma_I \rrbracket \, ds,$$
$$I_5 = \int_\Omega \nabla_h(\rho_h - \rho_I) \cdot (\sigma - \sigma_I)\, dx, \quad I_6 = \int_\Gamma \llbracket \rho_h - \rho_I \rrbracket \cdot \{\sigma - \sigma_I\}\, ds,$$
$$I_7 = \int_{\Gamma_0} \theta \cdot \llbracket \rho_h - \rho_I \rrbracket \llbracket \sigma - \sigma_I \rrbracket \, ds, \quad I_8 = \int_\Gamma \mu \llbracket \Pi_h(\rho - \rho_I) \rrbracket \cdot \llbracket \rho_h - \rho_I \rrbracket \, ds.$$

Clearly we have

$$I_1 \leq Ch|\sigma|_{1,\Omega}\|\sigma_h - \sigma_I\|_{0,\Omega}, \qquad I_2 \leq Ch|\rho|_{2,\Omega}\|\sigma_h - \sigma_I\|_{0,\Omega}.$$

Also, since ρ and ρ_I are continuous functions in Ω vanishing on $\partial\Omega$ we have $I_3 = I_4 = I_8 = 0$, and since $\nabla_h(\rho_h - \rho_I) \in P_0(\mathcal{T}_h)$, it results $I_5 = 0$.

Moreover, we can write I_6 as

$$I_6 = \int_\Gamma \llbracket \rho_h - \Pi_h \rho_h \rrbracket \cdot \{\sigma - \sigma_I\}\, ds + \int_\Gamma \llbracket \Pi_h \rho_h \rrbracket \cdot \{\sigma - \sigma_I\}\, ds =: A + B.$$

Applying the trace inequality (6.3), we find for B

$$B \leq C \sum_{e \in \mathcal{E}} \left\{ \| \llbracket \Pi_h \rho_h \rrbracket \|_{0,e} \sum_{K \in \mathcal{T}_h : e \subseteq K} \left(h_e^{-\frac{1}{2}} \|\sigma - \sigma_I\|_{0,K} + h_e^{\frac{1}{2}} |\sigma|_{1,K} \right) \right\}$$
$$\leq Ch^{\frac{1}{2}} |\sigma|_{1,\Omega} |\rho_h - \rho_I|_\star.$$

Using the inequalities (6.4) (with $q = \rho_h - \rho$) and (6.3), and taking into account again that $[\![\rho - \Pi_h \rho]\!] = 0$ on Γ (and the regularity of the mesh $\mathcal{T}_h$), we have

$$\begin{aligned} A &\leq \sum_{e\in\mathcal{E}} \| [\![\rho_h - \Pi_h \rho_h]\!] \|_{0,e} \|\{\sigma - \sigma_I\}\|_{0,e} \\ &\leq C \sum_{e\in\mathcal{E}} \left(h_K^{1/2} \sum_{K\in\mathcal{T}_h: e\subseteq K} h_K^{\frac{1}{2}} |\rho_h - \rho|_{1,K} \right) \left(\sum_{K\in\mathcal{T}_h: e\subseteq K} h_K^{\frac{1}{2}} |\sigma|_{1,K} \right) \\ &\leq Ch \left| h_{\mathcal{T}}^{\frac{1}{2}} (\rho - \rho_h) \right|_{1,h} |\sigma|_{1,\Omega}. \end{aligned}$$

Then, we need to estimate $|h_{\mathcal{T}}^{\frac{1}{2}}(\rho - \rho_h)|_{1,h}$. Since $\sigma = \nabla\rho$, and since σ_h is given by (3.8), we have

$$\nabla_h(\rho_h - \rho) = (\sigma_h - \sigma) - r([\![\rho_h]\!]) + l(\theta \cdot [\![\rho_h]\!]). \tag{6.8}$$

From (6.8) and Lemma 6.1, we obtain

$$\begin{aligned} \left| h_{\mathcal{T}}^{\frac{1}{2}} (\rho - \rho_h) \right|_{1,h} &\leq h^{\frac{1}{2}} \|\sigma - \sigma_h\|_{0,\Omega} + C|\rho_h|_\star \\ &\leq h^{\frac{1}{2}} \|\sigma - \sigma_I\|_{0,\Omega} + h^{\frac{1}{2}} \|\sigma_I - \sigma_h\|_{0,\Omega} + C|\rho_h|_\star \\ &\leq Ch^{\frac{3}{2}} |\sigma|_{1,\Omega} + h^{\frac{1}{2}} \|\sigma_I - \sigma_h\|_{0,\Omega} + C|\rho_I - \rho_h|_\star \end{aligned}$$

Hence

$$A \leq C(h^{\frac{5}{2}} |\sigma|_{1,\Omega}^2 + h^{\frac{3}{2}} \|\sigma_h - \sigma_I\|_{0,\Omega} |\sigma|_{1,\Omega} + h|\rho_I - \rho_h|_\star |\sigma|_{1,\Omega}).$$

Collecting the estimates for A and B, we get

$$I_6 \leq C(h^{\frac{5}{2}} |\sigma|_{1,\Omega}^2 + h^{\frac{3}{2}} \|\sigma_h - \sigma_I\|_{0,\Omega} |\sigma|_{1,\Omega} + h^{\frac{1}{2}} |\rho_I - \rho_h|_\star |\sigma|_{1,\Omega}).$$

Clearly, it also holds

$$I_7 \leq C(h^{\frac{5}{2}} |\sigma|_{1,\Omega}^2 + h^{\frac{3}{2}} \|\sigma_h - \sigma_I\|_{0,\Omega} |\sigma|_{1,\Omega} + h^{\frac{1}{2}} |\rho_I - \rho_h|_\star |\sigma|_{1,\Omega}).$$

Now we can go back to equation (6.7), obtaining, after applying the arithmetic-geometric inequality to the terms bounding the I_i's, $i = 1, \ldots, 8$, that

$$\|\sigma_h - \sigma_I\|_{0,\Omega}^2 + |\rho_h - \rho_I|_\star^2 \leq \frac{1}{2} \|\sigma_h - \sigma_I\|_{0,\Omega}^2 + \frac{1}{2} |\rho_h - \rho_I|_\star^2 + Ch|\rho|_{2,\Omega}^2,$$

from which it follows

$$\|\sigma_h - \sigma_I\|_{0,\Omega}^2 + |\rho_h - \rho_I|_\star^2 \leq Ch|\rho|_{2,\Omega}^2.$$

Now, the triangle inequality and standard interpolation estimates give us the assertion of the Theorem. □

Notice that Theorem 3.4 together with (6.8) and Lemma 6.1 give us

$$\left| h_{\mathcal{T}}^{\frac{1}{2}} (\rho - \rho_h) \right|_{1,h} \leq Ch^{\frac{1}{2}} |\rho|_{2,\Omega}. \tag{6.9}$$

Proof of Theorem 3.5. The proof of (3.21) follows by standard arguments [3] using Proposition 6.2 and taking into account that the numerical fluxes $\widehat{\rho}$ and $\widehat{\sigma}$ are conservative (that is, $[\![\widehat{\rho}]\!] = 0$ on Γ and $[\![\widehat{\sigma}]\!] = 0$ on Γ_0) and then the modified IP and LDG become adjoint consistent. We refer to [3] for details.

Now we prove (3.22) for the modified LDG method with μ given by (3.16). The proof follows from duality arguments similar to those used in [14]. Let w be the solution of

$$-\Delta w = \rho - \rho_h \qquad \text{in } \Omega$$
$$w = 0 \qquad \text{on } \partial\Omega,$$

and let $\widetilde{w}_h$ be the finite element approximation of w in the conforming piecewise linear space $\widetilde{V}_h = P_1(\mathcal{T}_h) \cap H_0^1(\Omega)$. We have $w \in H^2(\Omega)$,

$$\|w\|_{2,\Omega} \leq C\|\rho - \rho_h\|_{0,\Omega}, \tag{6.10}$$

and the error estimate

$$\|w - \widetilde{w}_h\|_{1,\Omega} \leq Ch|w|_{2,\Omega}. \tag{6.11}$$

Using that $-\Delta\rho = f$ and that ρ_h verifies $B_h(\rho_h, \widetilde{w}_h) = \int_\Omega f\widetilde{w}_h\, dx$ it is easy to check that

$$\int_\Omega \nabla_h(\rho - \rho_h) \cdot \nabla\widetilde{w}_h\, dx = -\int_\Gamma [\![\, \rho_h \,]\!] \cdot \{\nabla\widetilde{w}_h\}\, ds - \int_{\Gamma_0} [\![\, \rho_h \,]\!]\, \theta \cdot [\![\, \nabla\widetilde{w}_h \,]\!]\, ds.$$

Then

$$\begin{aligned}
\|\rho - \rho_h\|_{0,\Omega}^2 &= \sum_{K\in\mathcal{T}_h} -\int_K \Delta w(\rho - \rho_h)\, dx \\
&= \sum_{K\in\mathcal{T}_h} \left(\int_K \nabla(w - \widetilde{w}_h) \cdot \nabla_h(\rho - \rho_h)\, dx \right. \\
&\qquad\qquad \left. + \int_K \nabla\widetilde{w}_h \cdot \nabla_h(\rho - \rho_h)\, dx + \int_{\partial K} \frac{\partial w}{\partial n}\rho_h\, ds \right) \\
&= \int_\Omega \nabla(w - \widetilde{w}_h) \cdot \nabla_h(\rho - \rho_h)\, dx + \int_\Gamma [\![\, \rho_h \,]\!] \cdot \{\nabla_h(w - \widetilde{w}_h)\}\, ds \\
&\qquad\qquad + \int_{\Gamma_0} [\![\, \rho_h \,]\!]\, \theta \cdot [\![\, \nabla_h(w - \widetilde{w}_h) \,]\!]\, ds,
\end{aligned}$$

where we used the regularity of w.

Assuming that μ is given by (3.16) and that the mesh $\mathcal{T}_h$ is quasiuniform, we have

$$\left| \int_\Gamma [\![\, \rho_h \,]\!] \cdot \{\nabla_h(w - \widetilde{w}_h)\}\, ds + \int_{\Gamma_0} [\![\, \rho_h \,]\!]\, \theta \cdot [\![\, \nabla_h(w - \widetilde{w}_h) \,]\!]\, ds \right| \leq$$
$$C(\theta)|\rho - \rho_h|_\star (h^{-\frac{1}{2}}|w - \widetilde{w}_h|_{1,\Omega} + h^{\frac{1}{2}}|w|_{2,\Omega}),$$

and from (6.9) we have $|(\rho - \rho_h)|_{1,h} \leq C|\rho|_{2,\Omega}$. Then, using Theorem 3.4 and (6.10)–(6.11), we get

$$\|\rho - \rho_h\|_{0,\Omega}^2 \leq Ch|\rho|_{2,\Omega}|w|_{2,\Omega} \leq Ch|\rho|_{2,\Omega}\|\rho - \rho_h\|_{0,\Omega},$$

obtaining (3.22). □

7. Conclusion

We introduced new Discontinuous Galerkin schemes in mixed form for symmetric elliptic problems of second order with reduced connectivity with respect to the standard ones. We proved that the modifications in the choice of the approximation spaces (Σ_h is made of piecewise constant functions, while V_h is the usual $\mathcal{P}_1(\mathcal{T}_h)$) and in the penalization term (an L^2-projection on $\mathcal{P}_0(e)$ is introduced) do not spoil the error estimates. The modified IP scheme and the modified LDG scheme are discussed in details. Some other schemes in the family of DG methods reduce, in the present case, either to modifid IP or to modified LDG.

These methods are then used for designing new exponentially fitted schemes for advection dominated equations. When the transport term is irrotational and the diffusion coefficient is constant, it turns out that simple manipulations at the algebraic level are possible for passing from the symmetric to the non symmetric case (see Remark 4.2). The presented numerical tests show the good performances of the proposed schemes. The boundary layers are very well treated: the width of the numerical layers is the best that can be obtained with a numerical scheme on a quasi-uniform mesh (not adapted to the layers), and the outflow boundary conditions are correctly imposed. Also internal layers created by sharp behaviors of the flow and of the right hand side are well captured (see Test 3 in section 5). Instead, the method suffers of crosswind diffusion in the case when a discontinuity is transported from the boundary, unless the mesh is aligned to the flow.

We conclude pointing out that the general scheme (4.27) can be extended to the case of unstructured meshes with hanging nodes, as explored by the authors in a work in preparation.

Acknowledgments The authors are very grateful to Ilaria Perugia for the valuable discussions and for providing a code for the symmetric problem with standard DG schemes. The work of the first author was partially supported by ALPHA (Academic Cooperation Project between European Union and Latin America, *Partial Differential Equations in Industry and Engineering*), and by Agencia Nacional para la Promoción Científica y Tecnológica (ANPCyT), Argentina, grant PICT 17-33617.

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h	$\|u - u_h\|_{L^2(\Omega)}$	$\|\sigma - \sigma_h\|_{L^2(\Omega)^2}$
0.18898	0.94460	0.91423
0.16667	0.85298	0.83733
0.094491	0.67420	0.65928
0.083333	0.60813	0.59991
0.047246	0.47797	0.47171
0.041667	0.43096	0.42756
0.023623	0.33820	0.33578
0.020833	0.30489	0.30359

TABLE 1. Errors for Test 1.

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