The Inapproximability for the (0,1)-additive number

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Abstract

An *additive labeling* of a graph G is a function $\ell : V(G) \rightarrow \mathbb{N}$, such that for every two adjacent vertices v and u of G, $\sum_{w\sim v} \ell(w) \neq \sum_{w\sim u} \ell(w)$ ($x \sim y$ means that x is joined to y). An *additive number* of G, denoted by $\eta(G)$, is the minimum number k such that G has a additive labeling $\ell : V(G) \rightarrow \{1, \ldots, k\}$. An *additive choosability number* of a graph G, denoted by $\eta_\ell(G)$, is the smallest number k such that G has an additive labeling from any assignment of lists of size k to the vertices of G.

Seamone (2012) [21] conjectured that for every graph G, $\eta(G) = \eta_\ell(G)$. We give a negative answer to this conjecture and we show that for every k there is a graph G such that $\eta_\ell(G) - \eta(G) \geq k$.

A $(0,1)$-*additive labeling* of a graph G is a function $\ell : V(G) \rightarrow \{0,1\}$, such that for every two adjacent vertices v and u of G, $\sum_{w\sim v} \ell(w) \neq \sum_{w\sim u} \ell(w)$. A graph may lack any $(0,1)$-additive labeling. We show that it is **NP**-complete to decide whether a $(0,1)$-additive labeling exists for some families of graphs such as planar triangle-free graphs and perfect graphs. For a graph G with some $(0,1)$-additive labelings, the $(0,1)$-additive number of G is defined as $\eta_1(G) = \min_{\ell\in\Gamma} \sum_{v\in V(G)} \ell(v)$ where Γ is the set of $(0,1)$-additive labelings of G. We prove that given a planar graph that contains a $(0,1)$-additive labeling, for all $\varepsilon > 0$, approximating the $(0,1)$-additive number within $n^{1-\varepsilon}$ is **NP**-hard.

Key words: Additive labeling; additive number; lucky number; $(0,1)$-additive labeling; $(0,1)$-additive number; Computational complexity.
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1 Introduction

An *additive labeling* of a graph G which was introduced by Czerwiński et al. [9], is a function $\ell : V(G) \to \mathbb{N}$, such that for every two adjacent vertices v and u of G, $\sum_{w\sim v}\ell(w) \neq \sum_{w\sim u}\ell(w)$ ($x \sim y$ means that x is joined to y). An *additive number* of G, denoted by $\eta(G)$, is the minimum number k such that G has a additive labeling $\ell : V(G) \to \{1,\dots,k\}$. Initially, additive labeling is called a lucky labeling of G. The following important conjecture is proposed by Czerwiński et al. [9].

Conjecture 1 [**Additive Coloring Conjecture** [9]] *For every graph G, $\eta(G) \leq \chi(G)$.*

Czerwiński et al. also, considered the list version of above problem [9]. An *additive choosability number* of a graph G, denoted by $\eta_\ell(G)$, is the smallest number k such that G has an additive labeling from any assignment of lists of size k to the vertices of G. Czerwiński et al. [9] proved that if T is a tree, then $\eta_\ell(T) \leq 2$, and if G is a bipartite planar graph, then $\eta_\ell(G) \leq 3$. Seamone in his Ph.D dissertation posed the following conjecture about the relationship between additive number and additive choosability number [20, 21].

Conjecture 2 [**Additive List Coloring Conjecture** [20, 21]] *For every graph G, $\eta(G) = \eta_\ell(G)$.*

For a given connected graph G with at least two vertices, if no two adjacent vertices have the same degree, then $\eta(G) = 1$ and $\eta_\ell(G) > 1$. We show that not only there exists a counterexample for the above equality but also the difference between $\eta(G)$ and $\eta_\ell(G)$ can be arbitrary large.

Theorem 1 *For every k there is a graph G such that $\eta(G) \leq k \leq \eta_\ell(G)/2$.*

Chartrand et al. introduced another version of additive labeling and called it sigma coloring [8]. For a graph G, let $c : V(G) \to \mathbb{N}$ be a vertex labeling of G. If for every two adjacent vertices v and u of G, $\sum_{w\sim v} c(w) \neq \sum_{w\sim u} c(w)$, then c is called a *sigma coloring* of G. The minimum number of labels required in a sigma coloring is called the *sigma chromatic number* of G and is denoted by $\sigma(G)$. Chartrand et al. proved that, for every graph G, $\sigma(G) \leq \chi(G)$ [8].

Theorem A [8] *For every graph G, $\sigma(G) \leq \chi(G)$.*

Additive labeling and sigma coloring have been studied extensively by several authors, for instance see [3, 4, 7, 8, 9, 11, 14, 19]. It is proved, in [3] that it is **NP**-complete to determine whether a given graph G has $\eta(G) = k$ for any $k \geq 2$. Also, it was shown that, it is **NP**-complete to decide for a given planar 3-colorable graph G, whether $\eta(G) = 2$ [3]. Furthermore, it was proved that, it is **NP**-complete to decide for a given 3-regular graph G, whether $\eta(G) = 2$ [11].

The edge version of additive labeling was introduced by Karoński, Łuczak and Thomason [16]. They introduced an edge-labeling which is additive vertex-coloring that means for every edge uv, the sum of labels of the edges incident to u is different from the sum of labels of the edges incident to v [16]. It is conjectured that three integer labels $\{1, 2, 3\}$ are sufficient for every connected graph, except K_2 [16]. Currently the best bound is 5 [15]. This labeling has been studied extensively by several authors, for instance see [1, 2, 5, 17, 18].

A *clique* in a graph $G = (V, E)$ is a subset of its vertices such that every two vertices in the subset are connected by an edge. The *clique number* $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. There is no direct relationship between the additive number and the clique number of graphs. For any natural number n there exists a graph G, such that $\omega(G) = n$ and $\eta(G) = 1$. To see this for given number n, consider a graph G with the set of vertices $V(G) = \{v_i | 1 \leq i \leq n\} \cup \{u_{i,j} | 1 \leq j < i \leq n\}$ and the set of edges $E(G) = \{v_i v_j | i \neq j\} \cup \{v_i u_{i,j} | 1 \leq j < i \leq n\}$.

Theorem 2 *We have the following:*
(i) *For every graph G, $\eta(G) \geq \frac{w}{n-w+1}$.*
(ii) *If G is a regular graph and $\omega > \frac{n+4}{3}$, then $\eta(G) \geq 3$.*

A $(0, 1)$-*additive labeling* of a graph G is a function $\ell : V(G) \to \{0, 1\}$, such that for every two adjacent vertices v and u of G, $\sum_{w\sim v} \ell(w) \neq \sum_{w\sim u} \ell(w)$. A graph may lack any $(0, 1)$-additive labeling. It was proved that, it is **NP**-complete to decide for a given 3-regular graph G, whether $\eta(G) = 2$ [11]. So, it is **NP**-complete to decide whether a $(0, 1)$-additive labeling exists for a given 3-regular graph G. In this paper, we study the computational complexity of $(0, 1)$-additive labeling for planar graphs. We show that it is **NP**-complete to decide whether a $(0, 1)$-additive labeling exists for some families of graphs such as planar triangle-free graphs.

Theorem 3 *It is* **NP**-*complete to determine whether a given a planar triangle-free graph G has a $(0, 1)$-additive labeling?*

For a graph G with some $(0,1)$-additive labelings, the $(0,1)$-additive number of G is defined as $\eta_1(G) = \min_{\ell\in\Gamma} \sum_{v\in V(G)} \ell(v)$ where Γ is the set of $(0,1)$-additive labelings of G. For a given graph G with a $(0,1)$-additive labeling ℓ the function $1 + \sum_{v\in V(G)} \ell(v)$ is a proper vertex coloring, so we have the following trivial lower bound for $\eta_1(G)$.

$$\chi(G) - 1 \leq \eta_1(G).$$

We prove that given a planar graph that contains a $(0,1)$-additive labeling, for all $\varepsilon > 0$, approximating the $(0,1)$-additive number within $n^{1-\varepsilon}$ is **NP**-hard.

Theorem 4 *If* $\mathbf{P} \neq \mathbf{NP}$, *then for any constant* $\varepsilon > 0$, *there is no polynomial-time* $n^{1-\varepsilon}$-*approximation algorithm for finding* $\eta_1(G)$ *for a given planar graph with at least one* $(0,1)$-*additive labeling.*

A graph G is called *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. Finally, we show that it is **NP**-complete to decide whether a $(0,1)$-additive labeling exists for perfect graphs.

Theorem 5 *The following problem is* **NP**-*complete: Given a perfect graph* G, *does* G *have any* $(0,1)$-*additive labeling?*

For $v \in V(G)$ we denote by $N(v)$ the set of neighbors of v in G. Also, for every $v \in V(G)$, the degree of v is denoted by $d(v)$. We follow [13, 22] for terminology and notation not defined here, and we consider finite undirected simple graphs $G = (V, E)$.

2 Counterexample

Proof of Theorem 1. For every k we construct a graph G such that $\eta_\ell(G) - \eta(G) \geq k$. For every α, $1 \leq \alpha \leq 2k-1$ consider a copy of complete graph $K_{2k}^{(\alpha)}$, with the vertices $\{x_\beta^\alpha : 1 \leq \beta \leq k\} \cup \{y_\beta^\alpha : 1 \leq \beta \leq k\}$. Next, consider an isolated vertex t and join every vertex y_β^α to t, Call the resulting graph G. First, note that in every additive labeling ℓ of G, for every $1 \leq i < j \leq k$ we have $\sum_{z\in N(x_i^1)} \ell(z) \neq \sum_{z\in N(x_j^1)} \ell(z)$, thus $\ell(x_i^1) \neq \ell(x_j^1)$ (because all the neighbors of x_i^1 and x_j^1 are common except x_i^1 as a neighbor of x_j^1, and vice versa). Therefore $\ell(x_1^1), \ell(x_2^1), \ldots, \ell(x_k^1)$ are k distinct numbers, that means $\eta(G) \geq k$. Define:

$\ell : V(G) \to \{1, 2, \ldots, 2k\}$,

$\ell(x^{\alpha}_{\beta}) = \ell(y^{\alpha}_{\beta}) = \beta$, for every α and β,

$\ell(t) = k$.

It is easy to see that ℓ is an additive labeling for G. Next, we show that $\eta_\ell(G) > 2k-1$. Consider the following lists for the vertices of G.

$L(x^{\alpha}_{\beta}) = \{1, 2, 3, \ldots, 2k-1\}$, for every α and β,

$L(y^{\alpha}_{\beta}) = \{1+\alpha, 2+\alpha, 3+\alpha, \ldots, 2k-1+\alpha\}$, for every α and β,

$L(t) = \{1, 2, 3, \ldots, 2k-1\}$.

To the contrary suppose that $\eta_\ell(G) \leq 2k-1$ and let ℓ be an additive labeling from the above lists. Suppose that $\ell(t) = r$. Consider the complete graph $K^{(r)}_{2k}$, we have:

$L(x^{r}_{\beta}) = \{1, 2, 3, \ldots, 2k-1\}$, $1 \leq \beta \leq k$,

$L(y^{r}_{\beta}) = \{1+r, 2+r, 3+r, \ldots, 2k-1+r\}$, $1 \leq \beta \leq k$.

Now, consider the following partition for $\{1, 2, 3, \ldots, 2k-1\} \cup \{1+r, 2+r, 3+r, \ldots, 2k-1+r\}$,

$$\{1+r, 1\}, \{2+r, 2\}, \ldots, \{2k-1+r, 2k-1\}$$

By Pigeonhole Principle, there are indices i, n and m such that $\ell(x^{r}_{m}), \ell(y^{r}_{n}) \in \{i+r, i\}$, so $\ell(x^{r}_{m}) = i$ and $\ell(y^{r}_{n}) = i+r$. Therefore, $\sum_{z \in N(x^{r}_{m})} \ell(z) = \sum_{z \in N(y^{r}_{n})} \ell(z)$. This is a contradiction, so $\eta_\ell(G) \geq 2k$.

□

3 Lower bounds

Proof of Theorem 2. **(i)** Let $\ell : V(G) \to \{1, \ldots, k\}$ be an additive labeling of G and suppose that $T = \{v_1, \ldots, v_\omega\}$ is a maximum clique in G. For each vertex $v \in T$, define the function Y_v.

$$Y_v \stackrel{\text{def}}{=} \sum_{\substack{x \in V(G) \setminus T \\ x \sim v}} l(x) - l(v).$$

For every two adjacent vertices v and u in T, we have:

$$\sum_{x \sim v} l(x) \neq \sum_{x \sim u} l(x),$$

$$\sum_{\substack{x\notin T\\ x\sim v}} l(x) + \sum_{\substack{x\in T\\ x\neq v}} l(x) \neq \sum_{\substack{x\notin T\\ x\sim u}} l(x) + \sum_{\substack{x\in T\\ x\neq u}} l(x),$$
$$\sum_{\substack{x\notin T\\ x\sim v}} l(x) + l(u) \neq \sum_{\substack{x\notin T\\ x\sim u}} l(x) + l(v),$$
$$Y_v \neq Y_u.$$

Thus, $Y_{v_1}, \dots, Y_{v_\omega}$ are distinct numbers. On the other hand, for each vertex $v \in T$, the domain of the function Y_v is $[-k, k(n-w)-1]$. So $w \le k(n-w+1)$, therefore $k \ge \frac{w}{n-w+1}$ and the proof is completed.

(ii) Let G be a regular graph, obviously $\eta(G) \ge 2$. To the contrary suppose that $\eta(G) = 2$. Let T be a maximum clique in G and $c : V(G) \to \{1, 2\}$ be an additive labeling of G. Define:

$$X_1 = c^{-1}(1) \cap T, \qquad X_2 = c^{-1}(2) \cap T,$$
$$Y_1 = c^{-1}(1) \setminus T, \qquad Y_2 = c^{-1}(2) \setminus T.$$

Suppose that $X_1 = \{v_1, \dots, v_k\}$ and $X_2 = \{v_{k+1}, \dots, v_\omega\}$. For each $1 \le i \le \omega$, denote the number of neighbors of v_i, in Y_1 by d_i. Since c is an additive labeling of the regular graph, so every two adjacent vertices have different numbers of neighbors in $c^{-1}(1)$. Therefore $d_1, \dots, d_k, 1 + d_{k+1}, \dots, 1 + d_\omega$ are distinct numbers. Since for each $1 \le i \le \omega$, $0 \le d_i \le |Y_1|$, we have $|Y_1| \ge \omega - 2$. Similarly, $|Y_2| \ge \omega - 2$, so

$$n = |T| + |Y_1| + |Y_2| \ge 3\omega - 4.$$

This is a contradiction. So the proof is completed.

□

4 Planar graphs

Proof of Theorem 3. Let Φ be a 3-SAT formula with the set of clauses C and the set of variables X. Let $G(\Phi)$ be a graph with the vertices $C \cup X \cup (\neg X)$, where $\neg X = \{\neg x : x \in X\}$, such that for each clause $c = y \vee z \vee w$, c is adjacent to y, z and w, also every $x \in X$ is adjacent to $\neg x$. Φ is called planar 3-SAT(type 2) formula if $G(\Phi)$ is a planar graph. It was shown that the problem of satisfiability of planar 3-SAT(type 2) is **NP**-complete [12]. In order to prove our theorem, we reduce the following problem to our problem.

Problem: *Planar* 3-*SAT(type* 2*).*
INPUT: A 3-SAT(type 2) formula Φ.

Question: Is there a truth assignment for Φ that satisfies all the clauses?

Consider an instance of planar 3-SAT(type 2) with the set of variables X and the set of clauses C. We transform this into a graph $G'(\Phi)$ such that $G'(\Phi)$ has a $(0,1)$-additive labeling, if and only if Φ is satisfiable. The graph $G'(\Phi)$ has a copy of $B(x)$ for each variable x and a copy of $A(c)$ for each clause c. $B(x)$ and $A(c)$ are shown in Figure 1. Also, for every $c \in C$, $x \in X$, the edge $w_c^1 x$ is added if c contains the literal x. Furthermore, for every $c \in C$, $\neg x \in \neg X$, the edge $w_c^1 \neg x$ is added if c contains the literal $\neg x$. Call the resulting graph $G'(\Phi)$. Clearly $G'(\Phi)$ is triangle-free and planar.

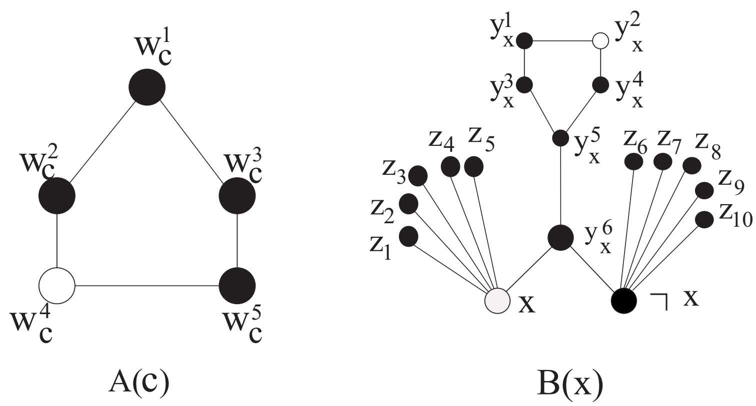

Figure 1: The two auxiliary graphs $A(c)$ and $B(x)$.

Fact 1 Let ℓ be a $(0,1)$-additive labeling for $G'(\Phi)$, for each clause $c = a \vee b \vee c$, $\ell(a) + \ell(b) + \ell(c) \geq 1$.

Proof of Fact 1. To the contrary suppose that there exists a clause $c = a \vee b \vee c$, such that $\ell(a) + \ell(b) + \ell(c) = 0$, then $\sum_{t \in N(w_c^1)} \ell(t) = \ell(w_c^2) + \ell(w_c^3)$. Consider the odd cycle $w_c^1 w_c^2 w_c^4 w_c^5 w_c^3$, but an odd cycle does not have any $(0,1)$-additive labeling, this is a contradiction. ♠

Fact 2 Let $G'(\Phi)$ be a graph with a $(0,1)$-additive labeling ℓ, for each variable x, $\ell(x) + \ell(\neg x) \leq 1$.

Proof of Fact 2. To the contrary, suppose that there is a variable x, such that $\ell(x) + \ell(\neg x) = 2$. Consider the auxiliary graph $B(x)$. Because of the odd cycle $y_x^1 y_x^2 y_x^4 y_x^5 y_x^3$, $\ell(y_x^6) = 1$. Now two cases for $\ell(y_x^5)$ can be considered.

Case 1. $\ell(y_x^5) = 1$. Thus $\sum_{t \in N(y_x^6)} \ell(t) = 3$, therefore $\sum_{t \in N(y_x^5)} \ell(t) \in \{1, 2\}$.

• If $\sum_{t \in N(y_x^5)} \ell(t) = 1$, then $\ell(y_x^3) = \ell(y_x^4) = 0$. Thus, $\ell(y_x^1) + \ell(y_x^2) = 1$; without loss of generality suppose that $\ell(y_x^1) = 1$ and $\ell(y_x^2) = 0$, in this case $\sum_{t \in N(y_x^2)} \ell(t) = \sum_{t \in N(y_x^4)} \ell(t)$, but this is a contradiction.

• If $\sum_{t \in N(y_x^5)} \ell(t) = 2$. Suppose that $\ell(y_x^3) = 1$, $\ell(y_x^4) = 0$. Four subcases for $\ell(y_x^1), \ell(y_x^2)$ can be considered, each of them produces a contradiction.

Case 2. $\ell(y_x^5) = 0$. Thus $\sum_{t \in N(y_x^6)} \ell(t) = 2$, therefore $\sum_{t \in N(y_x^5)} \ell(t) \in \{1, 3\}$.

• If $\sum_{t \in N(y_x^5)} \ell(t) = 1$, so $\ell(y_x^3) = \ell(y_x^4) = 0$. Therefore, $\ell(y_x^1) + \ell(y_x^2) = 1$. With no loss of generality suppose that $\ell(y_x^1) = 1$, $\ell(y_x^2) = 0$, therefore $\sum_{t \in N(y_x^3)} \ell(t) = \sum_{t \in N(y_x^5)} \ell(t)$, but this is a contradiction.

• If $\sum_{t \in N(y_x^5)} \ell(t) = 3$, so $\ell(y_x^3) + \ell(y_x^4) = 2$. So $\ell(y_x^1) + \ell(y_x^2) = 1$. Suppose that $\ell(y_x^1) = 1$, $\ell(y_x^2) = 0$, therefore $\sum_{t \in N(y_x^1)} \ell(t) = \sum_{t \in N(y_x^3)} \ell(t)$, this is a contradiction. ♠

First, suppose that Φ is satisfiable with the satisfying assignment $\Gamma : X \to \{true, false\}$. We present a $(0, 1)$-additive labeling ℓ for $G'(\Phi)$; for every variable x if $\Gamma(x) = true$, then put $\ell(x) = 1$, otherwise put $\ell(\neg x) = 1$. Also put $\ell(z_1) = \cdots \ell(z_{10}) = \ell(y_x^1) = \ell(y_x^3) = \ell(y_x^4) = \ell(y_x^5) = \ell(y_x^6) = 1$. Moreover, for every clause c, put $\ell(w_c^1) = \ell(w_c^2) = \ell(w_c^3) = \ell(w_c^5) = 1$. It is easy to extend this labeling to a $(0, 1)$-additive labeling for $G'(\Phi)$. Next, suppose that $G'(\Phi)$ has a $(0, 1)$-additive labeling ℓ. For each variable x, by Fact 2, $\ell(x) + \ell(\neg x) \leq 1$. If $\ell(x) = 1$, put $\Gamma(x) = true$, if $\ell(\neg x) = 1$, then put $\Gamma(x) = false$ and otherwise put $\Gamma(x) = true$. By Fact 1, Γ is a satisfying assignment for Φ. □

5 Inapproximability

Proof of Theorem 4. Let $\varepsilon > 0$ and k be a sufficiently large number. It was shown that 3-colorability of 4-regular planar graphs is **NP**-complete [10]. We reduce this problem to our problem, in more details for a given 4-regular planar graph G with k vertices, we construct a planar graph G^* with $7k + 10k^{\lceil \frac{3}{\varepsilon} \rceil + 2}$ vertices, such that if $\chi(G) \leq 3$, then $\eta_1(G^*) \leq 5k$, otherwise $\eta_1(G^*) > 5k^{\lceil \frac{3}{\varepsilon} \rceil + 1}$, therefore there is no θ-approximation algorithm for determining $\eta_1(G^*)$ for planar graphs, where:

$$
\begin{aligned}
\theta = \frac{Approximate\ Answer}{OPT} &> \frac{5k^{\lceil \frac{3}{\varepsilon} \rceil + 1}}{5k} \\
&= k^{\lceil \frac{3}{\varepsilon} \rceil} \\
&= \left(k^{\lceil \frac{3}{\varepsilon} \rceil + 3}\right)^{\frac{\lceil \frac{3}{\varepsilon} \rceil}{\lceil \frac{3}{\varepsilon} \rceil + 3}} \\
&\geq \left(7k + 10k^{\lceil \frac{3}{\varepsilon} \rceil + 2}\right)^{\frac{\lceil \frac{3}{\varepsilon} \rceil}{\lceil \frac{3}{\varepsilon} \rceil + 3}} \\
&\geq |V(G^*)|^{\frac{\lceil \frac{3}{\varepsilon} \rceil}{\lceil \frac{3}{\varepsilon} \rceil + 3}} \\
&\geq |V(G^*)|^{1-\varepsilon}
\end{aligned}
$$

In order to construct G^*, we use the auxiliary graphs $D(v)$ which is shown in Figure 2. Using simple local replacements, for every vertex v of G, put a copy of $D(v)$, and for every edge vu of G, join the vertex v of $D(v)$ to the vertex u of $D(v)$. Call the resulting graph G^*. First, suppose that G is not 3-colorable and let ℓ be a $(0,1)$-additive labeling for G^*. By the structure of $D(v)$ we have $\ell(v) = 1$ and $\ell(p_3) = 0$, so $\sum_{x \in N(v)} \ell(x) = 4 + \ell(p_4) + \ell(p_5) + \ell(p_6)$. Since G is not 3-colorable, so there exists a vertex v such that $\sum_{x \in N(v)} \ell(x) = 3$, therefore in the subgraph $D(v)$, $\ell(p_4) + \ell(p_5) + \ell(p_6) = 0$, so $\ell(p_5) = 0$. Consequently for every i, $1 \leq i \leq d$, in the subgraph $D(v)$, $\ell(v_i) + \ell(v_i') \geq 1$. So $\eta_1(G^*) > 5k^{\lceil \frac{3}{\varepsilon} \rceil + 1}$. Next, suppose that $\chi(G) \leq 3$. So G has a proper vertex coloring $c : V(G) \to \{1, 2, 3\}$. For every vertex v of G, if $c(v) = 1$ put $\ell(p_4) = \ell(p_6) = 0$ and $\ell(p_5) = 1$, else if $c(v) = 2$ let $\ell(p_4) = 0$ and $\ell(p_5) = \ell(p_6) = 1$ and if $c(v) = 3$ let $\ell(p_4) = \ell(p_5) = \ell(p_6) = 1$. It is easy to extend ℓ to a $(0,1)$-additive labeling for G^* such that $\eta_1(G^*) \leq 5k$.

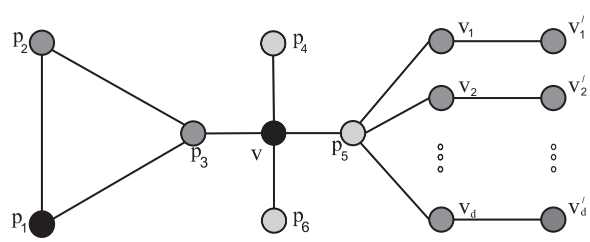

Figure 2: The auxiliary graph $D(v)$. This graph has $7 + 10k^{\lceil \frac{3}{\varepsilon} \rceil + 1}$ vertices, where $d = 5k^{\lceil \frac{3}{\varepsilon} \rceil + 1}$.

□

6 List Coloring Problem

Proof of Theorem 5. Let G be a graph and let L be a function which assigns to each vertex v of G a set $L(v)$ of positive integers, called the list of v. A proper vertex coloring $c : V(G) \to \mathbb{N}$ such that $f(v) \in L(v)$ for all $v \in V$ is called a *list coloring* of G with respect to L, or an *L-coloring*, and we say that G is *L-colorable*.

In the next, for a given graph G and a list $L(v)$ for every vertex v, we construct a graph H_G such that H_G has a $(0, 1)$-additive labeling if and only if G is L-colorable.

Define $W = \bigcup_{v \in V(G)} L(v)$ and let f be a bijective function from the set W to the set $\{2, 3, \cdots, |W| + 1\}$. For every vertex $v \in V(G)$, let $L_f(v) = \{f(i) | i \in L(v)\}$. G is L-colorable if and only if G is L_f-colorable. Now, we construct H_G form G and L_f.

Construction of H_G. We use three auxiliary graphs $T(w)$, $I(j)$ and $G(v, L_f(v), s)$. $T(w)$ and $I(j)$ are shown in Figure 3. Consider a vertex v and a copy of auxiliary graph $T(w)$. Join v to $T(w)$. Next, for every $j \in \{2, \ldots, s\} \setminus L_f(v)$ consider a copy of $I(j)$ and join v to u_j. Finally, put s isolated vertices and join each of them to v. Call the resulting graph $G(v, L_f(v), s)$. Now, for every vertex $v \in V(G)$ put a copy of $G(v, L_f(v), |W| + 1)$ and for every edge vv' in G join $v \in V(G(v, L_f(v), |W|+1)$ to $v' \in V(G(v', L_f(v'), |W|+1)$. Call the resulted graph H_G.

For a family $\mathscr{F}$ of graphs, define: $\mathscr{F}' \stackrel{\text{def}}{=} \{H_G | G \in \mathscr{F}\}$. We show that if $\mathscr{F}$ is a family of graphs such that *list coloring problem* is **NP**-complete over that family. Then, the following problem is **NP**-complete: "Given a graph $H_G \in \mathscr{F}'$, does H_G have a $(0, 1)$-additive labeling?

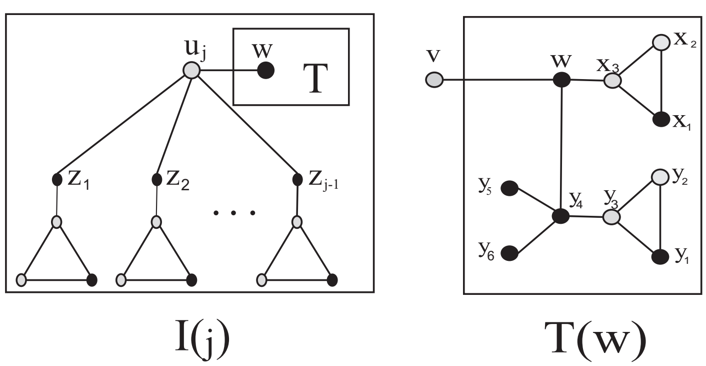

Figure 3: The auxiliary graphs $I(j)$ and $T(w)$.

First consider the following fact.

Fact 3 Let G be a graph with a $(0,1)$-additive labeling ℓ and have the auxiliary graph $T(w)$ as a subgraph, $\ell(v)=0$, $\ell(w)=1$ and $\sum_{x\in N(w)}\ell(x)=1$.

Proof of Fact 3. By attention to the two triangles $x_1x_2x_3$ and $y_1y_2y_3$, $\ell(w)=1$ and $\ell(y_4)=1$. Also $\ell(x_1)\neq\ell(x_2)$, without loss of generality suppose that $\ell(x_1)=1$ and $\ell(x_2)=0$. Therefore, $\ell(x_3)=0$, thus $\sum_{x\in N(w)}\ell(x)=1+\ell(v)$. Since $\sum_{x\in N(x_3)}\ell(x)=2$, therefore $\sum_{x\in N(w)}\ell(x)=1$, consequently $\ell(v)=0$. ♠

Fact 4 Let G be a graph with a $(0,1)$-additive labeling ℓ and have the auxiliary graph $I(j)$ as a subgraph, $\sum_{x\in N(u_j)}\ell(x)\geq j$.

Proof of Fact 4. By Fact 3, $\ell(w)=1$, while using similar arguments $\ell(z_1)=\cdots=\ell(z_{j-1})=1$. So $\sum_{x\in N(u_j)}\ell(x)\geq j$. ♠

Fact 5 Let ℓ be a $(0,1)$-additive labeling for $G(v,L_f(v),|W|+1)$, $\sum_{x\in N(v)}\ell(x)\in L_f(v)$.

Proof of Fact 5. By Fact 3 and Fact 4 it is clear.

First, suppose that H_G has a $(0,1)$-additive labeling ℓ, define $c:V(G)\rightarrow\mathbb{N}$, $c(v)=\sum_{x\in N(v)}\ell(x)$. c is a proper vertex coloring and for every vertex v, by Fact 5, $c(v)\in L_f(v)$. Next, suppose that G is L_f-colorable, then clearly, H_G has a $(0,1)$-additive labeling.

The list coloring problem is **NP**-complete for perfect graphs and planar graphs (see [6]). Obviously if G is a planar graph, then H_G is a planar graph. Also, if G is a perfect graph, then it is easy to see that H_G is a perfect graph. This completes the proof.

□

7 Concluding remarks

In this paper we study the computational complexity of $(0,1)$-additive labeling of graphs. A $(0,1)$-additive labeling of a graph G is a function $\ell:V(G)\rightarrow\{0,1\}$, such that for every two adjacent vertices v and u of G, $\sum_{w\sim v}\ell(w)\neq\sum_{w\sim u}\ell(w)$. We can consider another version of this problem that we call it proper total dominating set. *The proper total dominating set* of a graph $G=(V,E)$, that is a subset D of V such that every vertex has a neighbor in D (all vertices in the graph including the vertices in the dominating set have at least one neighbor in the dominating set) and every two adjacent vertices have a different number of neighbors in D.

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