Topological Additive Numbering of Directed Acyclic Graphs☆

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Abstract

We propose to study a problem that arises naturally from both Topological Numbering of Directed Acyclic Graphs, and Additive Coloring (also known as Lucky Labeling). Let D be a digraph and f a labeling of its vertices with positive integers; denote by $S(v)$ the sum of labels over all neighbors of each vertex v. The labeling f is called *topological additive numbering* if $S(u) < S(v)$ for each arc (u, v) of the digraph. The problem asks to find the minimum number k for which D has a topological additive numbering with labels belonging to $\{1, \ldots, k\}$, denoted by $\eta_t(D)$.

We characterize when a digraph has topological additive numberings, give a lower bound for $\eta_t(D)$ and provide an integer programming formulation for our problem. We also present some families for which $\eta_t(D)$ can be computed in polynomial time. Finally, we prove that this problem is $\mathcal{NP}$-Hard even when its input is restricted to planar bipartite digraphs.

Keywords: Additive coloring, Lucky labeling, Directed acyclic graphs, Topological numbering, Topological additive numbering, Computational complexity

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1. Introduction

Graph Coloring (GC) is one of the most representative problems in graph theory and combinatorial optimization because of its practical relevance and theoretical interest. Below, we present two known variants of GC.

Let $D = (V, A)$ be a directed acyclic graph (DAG), and let $S : V \to \mathbb{N}$ be a labeling of the vertices of D. If $S(u) < S(v)$ for every $(u, v) \in A$, then S is called a *topological numbering* of D [7]. We refer to the problem of finding the minimum number k for which such labeling S satisfies $S(v) \leq k$ for all $v \in V$ as *Topological Numbering* of DAGs (TN). This number k is also the size of the largest directed path in D (Gallai Theorem [9]). TN is solvable in polynomial time and generalizations of it give rise to different applications: PERT/CPM problems and the buffer assignment problem for weighted rooted graphs [5], and frequency assignment problems with fixed orientations [4].

The other variant of GC in which we are interested is *Additive Coloring* (AC), also known as Lucky Labeling. Let $G = (V, E)$ be a graph, $f : V \to \mathbb{N}$ a labeling of its vertices and $S(v)$ the sum of labels over all neighbors of v in G, i.e., $S(v) = \sum_{w \in N(v)} f(w)$, where $N(v)$ is the set of neighbors of v. If $S(u) \neq S(v)$ for every $(u, v) \in E$, then f is called *additive k-coloring* of G, where k is the largest label used in f. AC consists in finding the *additive chromatic number* of G, which is defined as the least number k for which G has an additive k-coloring and is denoted by $\eta(G)$.

AC was first presented by Czerwiński, Grytczuk and Zelazny [6]. They conjecture that $\eta(G) \leq \chi(G)$ for every graph G, where $\chi(G)$ is the chromatic number of G. The problem as well as the conjecture have recently gained considerable interest [1, 3, 10].

In particular, we proposed an exact algorithm for solving AC based on Benders' Decomposition [11]. This algorithm needs to solve several instances of an "oriented version" of AC. Let $D = (V, A)$ be a DAG, $f : V \to \mathbb{N}$ a labeling and $S(v) = \sum_{w \in N(v)} f(w)$ for all $v \in V$. If $S(u) < S(v)$ for every $(u, v) \in A$, then f is called *topological additive k-numbering* of D, with k the largest label used in f.

Unlike other coloring problems (including AC and TN), a digraph may lack any topological additive numbering. Let $\mathscr{D}$ denote the set of digraphs that have at least one topological additive numbering. Then, for $D \in \mathscr{D}$, the *topological additive number* of D, denoted by $\eta_t(D)$, is defined as the least number k for which D has a topological additive k-numbering. We call the problem of finding this number *Topological Additive Numbering* of DAGs (TAN).

As far as we know, there are no references to TAN in the literature. Our main contribution is to address TAN from a computational point of view. We first present some properties of TAN, including a lower bound for $\eta_t(D)$ and families of digraphs for which it is easy to exactly compute this number. At the end, we show that the problem is $\mathcal{NP}$-Hard even for planar bipartite digraphs.

2. Basic properties of TAN

Let $D = (V, A)$ be a DAG with $V = \{1, \ldots, n\}$. We will assume that D is connected, and its vertices are ordered so that $u < v$ holds whenever $(u, v) \in A$. As usual, $d(v)$ denotes the degree of vertex $v \in V$, and $G(D)$ the undirected underlying graph of D.

We first note that $\eta_t(D) \geq \eta(G(D))$. Therefore, lower bounds for the additive chromatic number also hold for the topological additive number. For instance, in [2] it is proved that $\eta(G(D)) \geq \lceil \omega/(n - \omega + 1) \rceil$, where ω is the size of a maximum clique of $G(D)$. However, it is possible to get a tighter bound for η_t as follows.

Proposition 1. *Let $D \in \mathscr{D}$, Q a clique of D and q_F, q_L the smallest and largest vertices of Q respectively. Then,*

$$\eta_t(D) \geq \left\lceil \frac{d(q_F) + 1}{d(q_L) - |Q| + 2} \right\rceil.$$

Proof. We follow [2]. Let f be a topological additive k-numbering of D. For each vertex $q \in Q$, let $Y_q = \sum_{w \in N(q) \setminus Q} f(w) - f(q)$. It is clear that $|N(q) \setminus Q| - k \leq Y_q \leq k|N(q) \setminus Q| - 1$.

On the other hand, for any $q_1, q_2 \in Q$ such that $q_1 < q_2$, we have $S(q_1) < S(q_2)$, or equivalently,

$$Y_{q_1} + \sum_{w \in Q} f(w) < Y_{q_2} + \sum_{w \in Q} f(w).$$

Hence, $Y_{q_1} < Y_{q_2}$. Since $q_F \leq q \leq q_L$ for all $q \in Q$, the values of Y_q must be between $|N(q_F) \setminus Q| - k$ and $k|N(q_L) \setminus Q| - 1$. By the pigeonhole principle, we obtain $|Q| \leq k|N(q_L) \setminus Q| - |N(q_F) \setminus Q| + k$. Therefore, $k \geq \lceil (d(q_F) + 1)/(d(q_L) - |Q| + 2) \rceil$. $\square$

Note that (i) this bound is tight for $D \in \mathscr{D}$ when $G(D)$ is a complete graph or a complete bipartite graph, and (ii) unlike the result given in [2], larger cliques do not necessarily lead to better lower bounds.

Now, we analyze when a digraph has topological additive numberings. The following is a sufficient condition.

Observation 1. *Let D be a DAG and u, v two vertices of D such that $N(u) \subseteq N(v)$. If there is a directed path from v to u, then $D \notin \mathscr{D}$.*

The previous condition is not necessary since the digraph in Figure 1 does not belong to $\mathscr{D}$ either.

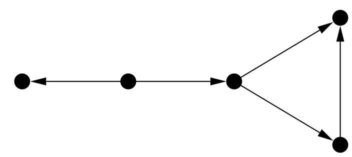

Figure 1: A digraph that does not belong to $\mathscr{D}$.

Although we do not know a combinatorial characterization of $\mathscr{D}$, we now describe a polynomial-time procedure that determines whether a digraph is in $\mathscr{D}$. Observe that the following integer linear program solves TAN:

$$
\begin{aligned}
&\min k \\
&\text{subject to} \\
&\qquad \sum_{w \in N(v)} f(w) - \sum_{w \in N(u)} f(w) \geq 1, && \forall \ (u,v) \in A \qquad (1)\\
&\qquad k - f(v) \geq 0, && \forall \ v \in V \qquad (2)\\
&\qquad f(v) \in \mathbb{N}, && \forall \ v \in V
\end{aligned}
$$

We call *IPF* this formulation and *LR* its linear relaxation, i.e., the linear program that comprises constraints (1), (2) and $f(v) \geq 1$ for all $v \in V$. If LR is infeasible, then $D \notin \mathscr{D}$. Otherwise, there exists an optimal solution of LR whose components are rational numbers; by multiplying these components by a suitable positive integer, we obtain a topological additive numbering of D. Therefore, LR is feasible if, and only if, $D \in \mathscr{D}$. Since deciding whether LR is feasible can be computed in polynomial time, we conclude that:

Proposition 2. *Given a DAG D, deciding whether $D \in \mathscr{D}$ is in $\mathcal{P}$.*

Next, we present some families of digraphs where TAN is solved in polynomial time. We say that a digraph D is *r-partite* when $G(D)$ is r-partite, and D is *complete* when $G(D)$ is complete. We say that an r-partite digraph is *monotone* when it can be partitioned into $V_1, V_2, \ldots, V_r$ and each of the arcs in $V_i \times V_j$ satisfies $i < j$. It is easy to see that a complete r-partite digraph belongs to $\mathscr{D}$ if, and only if, it is monotone. In this case, the topological additive number can be computed as follows.

Proposition 3. *Let D be a complete monotone r-partite digraph. Then,*

$$\eta_t(D) = \max\left\{ \left\lceil \frac{s_i}{|V_i|} \right\rceil : i = 1, \ldots, r \right\},$$

where $s_r = |V_r|$ and $s_i = \max\{1 + s_{i+1}, |V_i|\}$ for all $i = 1, \ldots, r-1$.

Proof. For any labeling f and set $S \subset V$, let $f(S) = \sum_{v \in S} f(v)$. Note that f is a topological additive numbering if, and only if, $f(V_i) > f(V_{i+1})$ for all $i = 1, \ldots, r-1$, since for all $j > i$, $u \in V_i$ and $w \in V_j$, we have $S(w) - S(u) = f(V_i) - f(V_j) > 0$.

Now, consider a labeling f such that, for all $i = 1, \ldots, r$, $f(V_i) = s_i$ and $f(v) \in \{\lfloor s_i/|V_i| \rfloor, \lceil s_i/|V_i| \rceil\}$ for all $v \in V_i$. Clearly, it is a topological additive p-numbering with $p = \max\{\lceil s_i/|V_i| \rceil : i = 1, \ldots, r-1\}$.

In order to prove that f is optimal, and by way of contradiction, suppose that there is a topological additive numbering f' such that $f'(V_j) < f(V_j)$ for some $j \in \{1, \ldots, r\}$; moreover, assume that j is the largest index satisfying this inequality. Then, from $f'(V_j) \geq |V_j|$ follows that

$$f'(V_j) < f(V_j) = 1 + s_{j+1} = 1 + f(V_{j+1}) \leq 1 + f'(V_{j+1}),$$

contradicting that $f'(V_j) > f'(V_{j+1})$. □

We now extend Proposition 3 for monotone (not necessarily complete) bipartite digraphs. As implied by Theorem 1 (in Section 3), it is $\mathcal{NP}$-hard to obtain $\eta_t(D)$ for general bipartite digraphs.

Proposition 4. *Let D be a monotone bipartite digraph. Then,*

$$\eta_t(D) = \max\left\{ \left\lfloor \frac{d(u)}{d(v)} \right\rfloor + 1 : v \in V_2, u \in N(v) \right\}.$$

Proof. Let $v^* \in V_2$ and $u^* \in N(v^*)$ be such that $\lfloor d(u^*)/d(v^*) \rfloor$ is maximized, and let $p = \lfloor d(u^*)/d(v^*) \rfloor + 1 = \lceil (d(u^*)+1)/d(v^*) \rceil$. Proposition 1 applied to $Q = \{u^*, v^*\}$ grants $\eta_t(D) \geq p$. A topological additive p-numbering f, defined by $f(v) = 1$ for vertices $v \in V_2$ and $f(v) = p$ for $v \in V_1$, provides the matching upper bound. $\square$

We end this section with a result that slightly extends the previous families. It can be used to remove universal vertices from a digraph under certain conditions, where a vertex v is *universal* when $N(v) = V \setminus \{v\}$.

Proposition 5. *Let $D = (V, A) \in \mathscr{D}$ without universal vertices, Δ be the largest degree in D, p be a positive integer such that $p \leq n - \Delta - 1$ and $D' = (V', A')$ be a digraph such that $V' = V \cup \{n+1, n+2, \ldots, n+p\}$ and $A' = A \cup \{(u, v) : u, v \in V', v > \max\{u, n\}\}$. Then, $\eta_t(D') = \max\{\eta_t(D), p\}$.*

Proof. Let f be an optimal topological additive numbering of D, and f' a labeling of D' satisfying $f'(v) = f(v)$ for all $v \in V$, and $f'(n+i) = p - i + 1$ for all $i = 1, \ldots, p$. In order to show f' is a topological additive numbering of D', let $P = p(p+1)/2$ and $Q = \sum_{w \in V'} f'(w)$. Now, for any $(u, v) \in A$, $S'(u) = P + S(u) < P + S(v) = S'(v)$; for any $u < v$ with $u, v \in V' \setminus V$, $S'(u) = Q - f'(u) < Q - f'(v) = S'(v)$; for any (u, v) with $u \in V$, $v \in V' \setminus V$, $S'(u) = Q - \sum_{w \in V \setminus N(u)} f(w) \leq Q - n + \Delta < Q - p \leq Q - f'(v) = S'(v)$.

In order to prove optimality, note first that no two vertices in V' can have the same label, implying that $\eta_t(D') \geq p$. Moreover, it is easy to see that if f' is a topological additive numbering of D', then f' restricted to the vertices of V is a topological numbering of D, implying that $\eta_t(D') \geq \eta_t(D)$. $\square$

3. Computational complexity of TAN

We have seen that deciding whether $D \in \mathscr{D}$ can be done in polynomial time. Moreover, deciding whether $\eta_t(D) = 1$ can be computed fast by checking whether $d(u) < d(v)$ for every arc (u, v). Nevertheless, deciding whether $\eta_t(D) = 2$ is $\mathcal{NP}$-complete. The proof given below shares the same approach of [1].

Let Φ be a 3-SAT formula with sets of clauses C and variables X; let $G_\Phi = (V_\Phi, E_\Phi)$ be the graph of Φ, where $V_\Phi = C \cup X \cup \{\neg x : x \in X\}$ and $E_\Phi = \{(x, \neg x) : x \in X\} \cup \{(c, y), (c, z), (c, w) : c \in C, c = y \vee z \vee w\}$. It is known that, given a 3-SAT formula Φ for which G_Φ is planar, deciding whether there is a truth assignment that satisfies Φ is $\mathcal{NP}$-complete [8].

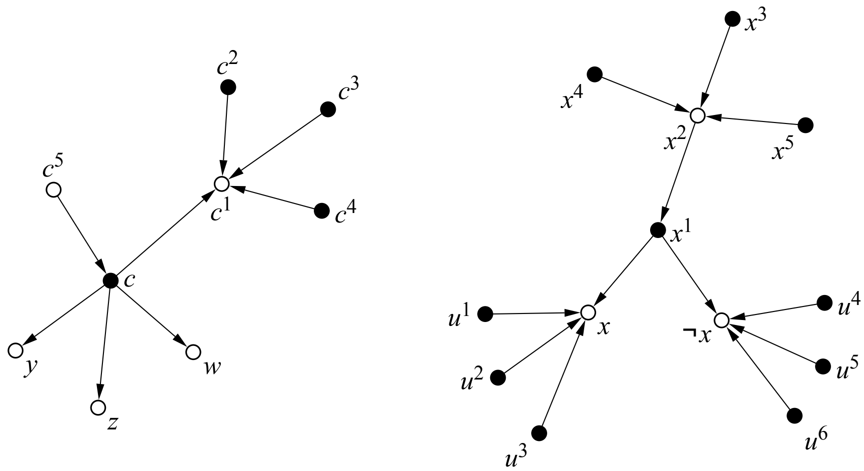

Figure 2: Construction of digraph D_Φ: for each variable x, D_Φ has a copy of the right digraph, and for each clause $c = y \vee z \vee w$, D_Φ has a copy of the left digraph. A bipartition is shown through the color of the vertices.

This problem is called *Planar 3-SAT (type 2)* (P3SAT2). We will assume, without loss of generality, that no literal is repeated within a clause (since, for instance, each clause of the form $y \vee y \vee z$ may be replaced by two clauses $x \vee y \vee z$ and $\neg x \vee y \vee z$, where x is an unused literal, maintaining planarity).

Our proof relies on a polynomial-time reduction from P3SAT2 to TAN. Consider an instance Φ of P3SAT2 and construct the following digraph D_Φ from G_Φ as follows (Figure 2):

- For each $x \in X$, add vertices $x^1, x^2, \ldots, x^5, u^1, u^2, \ldots, u^6$ to V, and replace edge $(x, \neg x)$ with arcs (x^1, x), $(x^1, \neg x)$, (x^2, x^1), (x^3, x^2), (x^4, x^2), (x^5, x^2), (u^1, x), (u^2, x), (u^3, x), $(u^4, \neg x)$, $(u^5, \neg x)$, $(u^6, \neg x)$.

- For each $c = y \vee z \vee w \in C$, add vertices $c^1, c^2, \ldots, c^5$ to V, and replace edges (c, y), (c, z) and (c, w) with arcs (c, y), (c, z), (c, w), (c, c^1), (c^2, c^1), (c^3, c^1), (c^4, c^1), (c^5, c).

By construction and since G_Φ is planar, $G(D_\Phi)$ is planar and bipartite.

For the next two lemmas assume that D_Φ has a topological additive 2-numbering f.

Lemma 1. *$f(x) + f(\neg x) \geq 3$ for all $x \in X$.*

Proof. In first place, $S(x^2) < S(x^1)$. Since x^2 has 4 neighbors, $S(x^2) \geq 4$ and then $S(x^1) \geq 5$. Since $S(x^1) = f(x) + f(\neg x) + f(x^2)$ and $f(x^2) \leq 2$, we get $f(x) + f(\neg x) \geq 3$. $\square$

Lemma 2. *$f(y) + f(z) + f(w) \leq 5$ for all $c = y \vee z \vee w \in C$.*

Proof. In first place, $S(c) < S(c^1)$. Since c^1 has 4 neighbors, $S(c^1) \leq 8$. Hence, $S(c) \leq 7$. Since $S(c) = f(y) + f(z) + f(w) + f(c^1) + f(c^5)$ and $f(c^1) + f(c^5) \geq 2$, we get $f(y) + f(z) + f(w) \leq 5$. $\square$

Theorem 1. *It is $\mathcal{NP}$-complete to decide whether $\eta_t(D) = 2$ for a digraph D whose underlying graph is planar and bipartite.*

Proof. We follow [1]. Let Φ be a 3-SAT formula such that G_Φ is planar, and D_Φ the digraph generated from G_Φ with the procedure given above. We only need to show that there exists a topological additive 2-numbering f of D_Φ *if and only if* there also exists a truth assignment $\Gamma : X \to \{true, false\}$ that satisfies Φ.
$\Leftarrow$) Let Γ be a truth assignment that satisfies Φ. Below, we propose a topological additive 2-numbering f of D_Φ:

- For each $x \in X$, let $f(x^1) = f(x^3) = f(x^4) = f(x^5) = 1$ and $f(x^2) = f(u^1) = f(u^2) = f(u^3) = f(u^4) = f(u^5) = f(u^6) = 2$; if $\Gamma(x) = true$ then let $f(x) = 1$ and $f(\neg x) = 2$, otherwise, let $f(x) = 2$ and $f(\neg x) = 1$. Then, $S(x^3) = S(x^4) = S(x^5) = 2$, $S(x^2) = 4$, $S(x^1) = 5$, $S(u^1) = S(u^2) = S(u^3) \leq 2$, $S(u^4) = S(u^5) = S(u^6) \leq 2$ and for all $x \in X \cup \neg X$ we have $S(x) \geq 7$. Moreover, $S(x) \geq 9$ when $(c, x) \in A$.

- For each $c \in C$, let $f(c) = f(c^2) = f(c^3) = f(c^4) = 2$ and $f(c^1) = f(c^5) = 1$. Then, $S(c^2) = S(c^3) = S(c^4) = 1$, $S(c^5) = 2$, $S(c^1) = 8$ and $5 \leq S(c) \leq 7$ (since Γ satisfies Φ).

$\Rightarrow$) Let f be a topological additive 2-numbering f of D_Φ. By Lemma 1, for each $x \in X$, the values $f(x)$ and $f(\neg x)$ cannot be both 1. Hence, we can set $\Gamma(x) = true$ when $f(x) = 1$ and $\Gamma(x) = false$ when $f(\neg x) = 1$. In the case that $f(x) = f(\neg x) = 2$, $\Gamma(x)$ may be arbitrarily *true* or *false*. Now, by Lemma 2, for every $c = y \vee z \vee w$, at least one of the three values $f(y)$, $f(z)$, $f(w)$ must be 1. Therefore, the assignment satisfies c and then Φ. $\square$

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