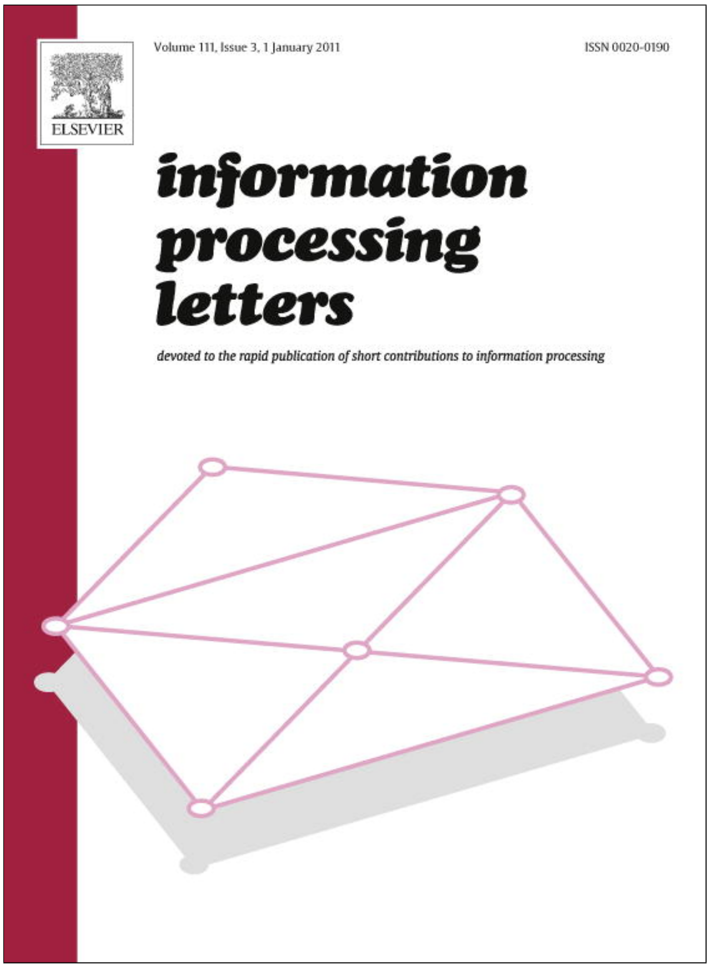
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The multiple domination and limited packing problems in graphs

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ABSTRACT

In this work we confront—from a computational viewpoint—the Multiple Domination problem, introduced by Harary and Haynes in 2000 among other variations of domination, with the Limited Packing problem, introduced in 2009. In particular, we prove that the Limited Packing problem is NP-complete for split graphs and for bipartite graphs, two graph classes for which the Multiple Domination problem is also NP-complete (Liao and Chang, 2003). For a fixed capacity, we prove that these two problems are polynomial time solvable in quasi-spiders. Furthermore, by analyzing the combinatorial numbers that are involved in their definitions applied to the join and the union of graphs, we show that both problems can be solved in polynomial time for P_4-tidy graphs. From this result, we derive that they are polynomial time solvable in P_4-lite graphs, giving in this way an answer to a question stated by Liao and Chang on the domination side.

1. Preliminaries and notation

Graphs in this work are simple and connected and for a graph G, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. A graph is *trivial* if it has at most one vertex.

For $v \in V(G)$, $N_G[v]$ denotes the *closed neighborhood* and $d_G(v)$, the *degree* of v in G. For $S \subseteq V(G)$, $N_G[S] := \bigcup_{v\in S} N_G[v]$. The minimum (maximum) degree between all the vertices in G is denoted by $\delta(G)$ ($\Delta(G)$).

Given two graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, the (*disjoint*) *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{ij\colon\ i \in V(G_1),\ \ j \in V(G_2)\}$.

A *stable set* of G is a set of pairwise nonadjacent vertices. The *stable set* problem consists of finding a stable set of G with maximum size. A *clique* is a set of pairwise adjacent vertices. A *vertex cover* is a subset of $V(G)$ that contains at least one endpoint of every edge. The *vertex cover* problem consists of finding a vertex cover of G with minimum size. These two problems are known to be NP-complete.

A graph G is *split* if $V(G)$ can be partitioned into a clique Q and a stable set S.

Among the variations of domination in graph theory, the k-tuple domination was introduced in [9]. On the other hand, k-limited packings were introduced in [6].

Given a graph G and a nonnegative integer k, $B \subseteq V(G)$ is a *k-limited packing* of G if $|N_G[v] \cap B| \leqslant k$, for every $v \in V(G)$. $L_k(G)$ denotes the cardinality of a k-limited packing of G of maximum size. It is clear that $L_k(G) \leqslant |V(G)|$ and $L_k(G) = |V(G)|$ if and only if $k \geqslant \Delta(G) + 1$. Observe that $L_0(G) = 0$ for every graph G.

The *Limited Packing* problem is formulated as

INSTANCE: A graph G; positive integers k and α.
QUESTION: Does G contain a k-limited packing of size at least α?

For a fixed positive integer k, the *k-Limited Packing* problem is formulated as

INSTANCE: A graph G; a positive integer α.
QUESTION: Does G contain a k-limited packing of size at least α?

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It is not difficult to see that both problems are in NP. Notice that their answers are nontrivial on instances with $\alpha \leqslant |V(G)| - 1$ and $|\Delta(G)| \geqslant k$.

Given a graph G and a nonnegative integer k, $D \subseteq V(G)$ is a *k-tuple dominating set* of G if $|N_G[v] \cap D| \geqslant k$, for every $v \in V(G)$. Notice that G has a k-tuple dominating sets if and only if $k \leqslant \delta(G)+1$ and, if G has a k-tuple dominating set D, then $|D| \geqslant k$. When $k \leqslant \delta(G)+1$, $\gamma_k(G)$ denotes the cardinality of a k-tuple dominating set of G of minimum size and $\gamma_k(G) = +\infty$, when $k > \delta(G)+1$. Observe that $\gamma_0(G) = 0$ for every graph G.

The *Multiple Domination* problem is formulated as

INSTANCE: A graph G; positive integers k and α.
QUESTION: Does G contain a k-tuple dominating set of size at most α?

For a positive integer k, the *k-Tuple Domination* problem is formulated as

INSTANCE: A graph G; a positive integer α.
QUESTION: Does G contain a k-tuple dominating set of size at most α?

It is not difficult to see that both problems are in NP. Notice that their answers are nontrivial on instances with $\alpha \geqslant k+1$ and $\delta(G)+1 \geqslant k$.

Two separate linear-time algorithms for solving the Multiple Domination and Limited Packing problems in tree graphs were provided independently (see [3,17]).

On the domination side, the authors extended the results in [17] by providing a linear-time algorithm for strongly chordal graphs [18].

Let us point out that, when a graph G is strongly chordal, the incidence matrix $N[G]$ of the closed neighborhoods of the vertices of G is *totally balanced* [5], i.e. the only square submatrix with two ones per row and per column is the 2×2 submatrix of all ones. A $0,1$ matrix is in *standard greedy* form if it contains no 2×2 submatrix of the form $\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$, where the order of the rows and columns is the same in the submatrix as in the matrix. The rows and columns of a totally balanced matrix of order $m \times n$ can be permuted into standard greedy form in time $O(nm^2)$ [1,12,19]. If $N[G]$ is in standard greedy form, the linear program $\max\{\sum_{i=1}^{m} y_i\colon\ yN[G] \leqslant k,\ 0 \leqslant y \leqslant 1\}$ can be solved by a greedy algorithm, which gives an integral optimal solution (see for example [2]). Then, when G is a strongly chordal graph, the objective $L_k(G)$ can be obtained by solving the previous linear program.

Concerning NP-completeness results, the authors in [18] proved that the Multiple Domination problem is NP-complete for split graphs and for bipartite graphs, and left open its complexity for other subclasses of perfect graphs.

Firstly, in this paper we prove that the Limited Packing problem is NP-complete for split graphs and for bipartite graphs. Then, we provide new graph classes where both problems—the Multiple Domination and Limited Packing problems—are polynomial time solvable; in particular, another class of perfect graphs, answering in this way a question stated by Liao and Chang [18] on the domination side.

Some of the results in this work have been already published—without proofs—in an electronic version [4].

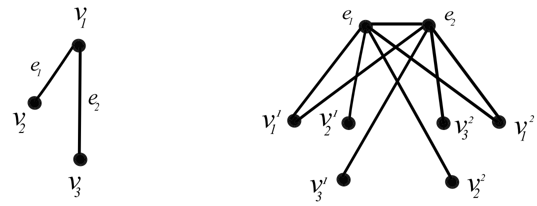

Fig. 1. G and G' in Theorem 1, for $k=2$.

The paper is organized as follows. Section 2 is devoted to the NP-completeness results. In Section 3, we show that the k-Limited Packing and k-Tuple Domination problems are polynomial time solvable in quasi-spiders. In Section 4, by analyzing the combinatorial numbers involved, we prove that they are also polynomial time solvable in P_4-tidy graphs. In addition, we find the computational complexity of the k-Tuple Domination problem for another subclass of perfect graphs.

2. The Limited Packing problem in split graphs and in bipartite graphs

In order to prove that the k-Tuple Domination problem is NP-complete for split graphs and for bipartite graphs, the authors in [18] reduced the vertex cover problem in both cases.

In this section, we will first reduce polynomially the stable set problem to the k-Limited Packing problem in a split graph. Actually, we have:

Theorem 1. *The k-Limited Packing problem is NP-complete for split graphs.*

Proof. We already know that the k-Limited Packing problem is in NP.

Let G be a graph with $V(G) = \{v_j\colon\ j = 1,\dots,n\}$ and $E(G) = \{e_l\colon\ l = 1,\dots,m\}$. We construct a split graph G' with $V(G') := E(G) \cup S$, where $S := \bigcup_{i=1}^{k} V^i$ and $V^i = \{v_j^i\colon\ j = 1,\dots,n\}$ for $i = 1,\dots,k$ and adjacencies defined in the following way: if $e = v_p v_q \in E(G)$, $N_{G'}[e] = \bigcup_{i=1}^{k} \{v_p^i, v_q^i\} \cup E(G)$ and if $v_j^i \in V^i$, $N_{G'}[v_j^i] = \{e \in E(G)\colon\ v_j$ is an extreme of $e \in G\}$, for $i = 1,\dots,k$ (see Fig. 1 for an example).

We will prove that G has a stable set of size s if and only if G' has a k-limited packing of size ks.

Given I, a stable set of G, it is straightforward that $\bigcup_{i=1}^{k} \{v_j^i\colon\ v_j \in I\}$ is a k-limited packing of G' and the only if part follows.

Now take B', a k-limited packing of G' of size ks.

Firstly observe that, if $e = v_p v_q \in B' \cap E(G)$, there exists $i \in \{1,\dots,k\}$ such that $v_p^i, v_q^i \notin B'$ and moreover, $B' \setminus \{e\} \cup \{v_p^i\}$ is also a k-limited packing of G'.

Secondly, if $e = v_p v_q \in E(G)$ and for some $i \in \{1,\dots,k\}$, $\{v_p^i, v_q^i\} \subseteq B'$, then $k \geqslant 2$ since v_p^i and v_q^i are both adjacent to e in G'. In this case, there exists $j \in \{1,\dots,k\}$ with $j \neq i$ such that $v_p^j, v_q^j \notin B'$. Thus $B' \setminus \{v_p^i\} \cup \{v_p^j\}$ is a k-limited packing of G'.

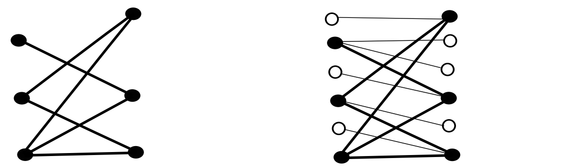

Fig. 2. G and G' in Theorem 2.

Thus, w.l.o.g. we can assume that $B' \cap E(G) = \emptyset$ and that $v_p^i \notin B'$ or $v_q^i \notin B'$, for every $i \in \{1, \ldots, k\}$ and every $e = v_p v_q$.

Consequently, for each $j = 1, \ldots, n$, the subset $\{v_j\colon v_j^i \in B'\}$ is a stable set of G of size $|B' \cap V^i|$, for $i = 1, \ldots, k$. Since $|B'| = ks$, there exists $i \in \{1, \ldots, k\}$ with $|B' \cap V^i| \geqslant s$. $\square$

For bipartite graphs, we have:

Theorem 2. *The Limited Packing problem is NP-complete for bipartite graphs.*

Proof. We already know that the Limited Packing problem is in NP.

We will reduce the Multiple Domination problem in a bipartite graph to the Limited Packing problem in a bipartite graph. Let G be a bipartite graph, k and α two positive integers, with $k \leqslant \delta(G)+1$ and $\alpha \geqslant k+1$, that define an instance of the Multiple Domination problem. We construct a graph G' such that $V(G') = V(G) \cup \bigcup_{v \in V(G)} S_v$, where S_v is a set on $\Delta(G) - d_G(v)$ new vertices, for each $v \in V(G)$, and $E(G') = E(G) \cup \bigcup_{v \in V(G)} \{vx\colon x \in S_v\}$ (see Fig. 2, where the vertices in white correspond to those in the stable sets S_v, for $v \in V(G)$). Clearly, the graph G' is bipartite and $G' = G$ when G is regular. Besides, let $k' = \Delta(G) - k + 1$.

We will prove that G has a k-tuple dominating set of size at most α if and only if G' has a k'-limited packing of size at least $\alpha' := |V(G')| - \alpha$.

Let R be a k-tuple dominating set of G with $|R| \leqslant \alpha$. We will demonstrate that $B := V(G') \setminus R$ is a k'-limited packing of G'. Clearly from its definition, $S_v \subseteq B$, for every $v \in V(G)$.

Take $w \in V(G')$. If $w \in S_v$ for some $v \in V(G)$, $|N_{G'}[w] \cap B|$ is equal to 1 or 2. In any case, we have $|N_{G'}[w] \cap B| \leqslant k'$. Indeed,

- when $k = \Delta(G)+1$—in which case G is regular—$k' = 0$, $R = V(G)$ and $B = \emptyset$;
- when $k = \Delta(G)$—in which case $d_G(x) \in \{\Delta(G) - 1, \Delta(G)\}$ for all $x \in V(G)$, $d_G(v) = \Delta(G) - 1$ and $N_G[v] \subseteq R$—$|N_{G'}[w] \cap B| = 1 = k'$;
- when $k \leqslant \Delta(G) - 1$, $k' \geqslant 2$.

If $w \in V(G') \cap V(G)$,

$$\begin{aligned} \left|N_{G'}[w] \cap B\right| &= |S_w \cap B| + \left|N_G[w] \setminus R\right| \\ &= |S_w| + \left|N_G[w] \setminus R\right| \\ &= \Delta(G) - d_G(w) + \left|N_G[w] \setminus R\right|. \end{aligned}$$

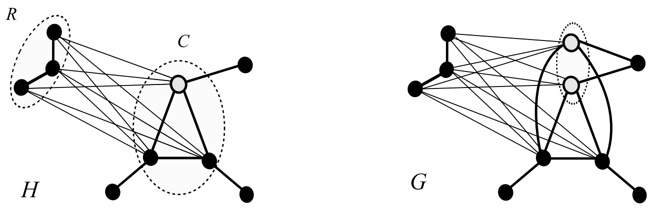

Fig. 3. A quasi-spider obtained from a thin spider. The white vertex in C is replaced by an S_2.

Since $|N_G[w] \cap R| \geqslant k$, $|N_G[w] \setminus R| \leqslant d_G(w) + 1 - k$, thus $|N_{G'}[w] \cap B| \leqslant k'$. Consequently, B is a k'-limited packing of G' and clearly holds $|B| = |V(G')| - |R| \geqslant \alpha'$.

To see the converse, let B be a k'-limited packing of G' with $|B| \geqslant \alpha'$ and let $R := V(G) \setminus B$. We will prove that R is a k-tuple dominating set of G.

Notice that, since $k \leqslant \delta(G) + 1$, $|S_v| = \Delta(G) - d_G(v) \leqslant \Delta(G) - k + 1 = k'$ for every $v \in V(G)$. W.l.o.g. we assume that $S_v \subseteq B$ for every $v \in V(G)$.

Then, given $v \in V(G)$, $|N_G[v] \cap B| = |N_{G'}[v] \cap B| - |B \cap S_v| = |N_{G'}[v] \cap B| - |S_v| = |N_{G'}[v] \cap B| - \Delta(G) + d_G(v)$. Since B is a k'-limited packing of G', we have $|N_G[v] \cap B| \leqslant d_G(v) + 1 - k$. This implies that $|N_G[v] \cap R| = d_G(v) + 1 - |N_G[v] \cap B| \geqslant k$, that is, R is a k-tuple dominating set of G. Clearly, $|R| = |V(G)| - |B| \leqslant \alpha$. $\square$

3. *k*-limited packings and *k*-tuple dominating sets of quasi-spiders

A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_r\}$ is a stable set, $C = \{c_1, \ldots, c_r\}$ is a clique, $r \geqslant 2$ and the *head* R is allowed to be empty. Moreover, all vertices in R are adjacent to all vertices in C and non-adjacent to all vertices in S. In a *thin spider*, s_i is adjacent to c_j if and only if $i = j$, and in a *thick spider*, s_i is adjacent to c_j if and only if $i \neq j$. It is straightforward that the complementary graph of a thin spider is a thick spider, and vice-versa. The triple (S, C, R) is called the (*spider*) *partition* and can be found in linear time [13].

A graph is a *quasi-spider* [7] if it is obtained from a spider with partition (S, C, R) by replacing at most one vertex of $S \cup C$ by a K_2 (clique on two vertices) or a S_2 (stable set on two vertices). Observe that a spider is an example of a quasi-spider. For the sake of simplicity, in the remainder of this work the partition of a quasi-spider obtained from a spider with partition (S, C, R) will be denoted by

$$(S \hookleftarrow W, C, R) \quad \text{or} \quad (S, C \hookleftarrow W, R),$$

meaning that some vertex of S or C was replaced by W, where W is a K_2, S_2, or the empty set in case the quasi-spider is a spider. When $W \neq \emptyset$, the vertex set of W will be denoted by $\{w_1, w_2\}$.

Besides, when G is a quasi-spider with partition $(S \hookleftarrow W, C, R)$ ($(S, C \hookleftarrow W, R)$), we will have the license to write $G = (S \hookleftarrow W, C, R)$ ($G = (S, C \hookleftarrow W, R)$) and to consider $S \subseteq (S \hookleftarrow W)$ and $C \subseteq (C \hookleftarrow W)$. (See Fig. 3.)

In this section we obtain closed formulas for the numbers $L_k(G)$ and $\gamma_k(G)$ when G is a quasi-spider.

Recall that for every graph G, $L_k(G) = |V(G)|$ when $k \geqslant \Delta(G) + 1$. For the remaining values of k, we have:

Proposition 3. *Let $H = (S, C, R)$, G be a quasi-spider obtained from H and $1 \leqslant k \leqslant \Delta(G)$.*

1. *If H is thin, then $L_k(G) = |S| + k - 1$.*
2. *If H is thick, then*

$$L_k(G) = \begin{cases} k & \text{when } 1 \leqslant k \leqslant |S| - 2, \\ k+1 & \text{when } |S| - 1 \leqslant k \leqslant \Delta(G). \end{cases}$$

Proof. 1. Suppose H is thin. Let us first show that $L_k(G) \geqslant |S| + k - 1$, for every $k \geqslant 1$. Take $A \subseteq R \cup C$ with $k - 1$ vertices, and define $B := S \cup A$. It is clear that $|N_G[v] \cap B| \leqslant k$ for each $v \in V(G)$, i.e. B is a k-limited packing of G, thus $L_k(G) \geqslant |S| + k - 1$.

To prove the other inequality, take B a k-limited packing of G. Let us remark that, if $B \subseteq R \cup C$ then $|B| \leqslant k$, since there exists $x \in C$ such that $R \cup C \subseteq N_G[x]$. Thus, when B is maximum, $B \cap S \neq \emptyset$, and consequently $|B \cap (R \cup C)| \leqslant k - 1$. Moreover, $S \subseteq B$ and therefore, $L_k(G) \leqslant |S| + k - 1$.

2. Suppose H is thick. Firstly, let us point out that any subset of $V(G)$ with k vertices is a k-limited packing of G, thus $L_k(G) \geqslant k$. Take $1 \leqslant k \leqslant |S| - 2$. Notice that there exists $x \in C$ such that $|N_G[x]| = |V(G)| - 1$. Then, given $T \subseteq V(G)$ with $|T| = k + 1$, there exists a vertex whose closed neighborhood contains T. This implies that $L_k(G) \leqslant k$. Therefore $L_k(G) = k$.

Now take $k \geqslant |S| - 1$. We will show that $B := S \cup B'$, with $B' \subseteq V(G) \setminus S$ and $|B'| = k - (|S| - 1)$ is a maximum k-limited packing of G.

- When $G = H$ or $G = (S, C \hookleftarrow W, R)$, take $B' \subseteq V(G) \setminus S$ with $|B'| = k - (|S| - 1)$. It holds that $|N_G[v] \cap B| \leqslant |S| - 1 + k - (|S| - 1) = k$ for every $v \in V(G)$, i.e. B is a k-limited packing of G and therefore $L_k(G) \geqslant k + 1$. Moreover, any subset of $V(G)$ with $k + 2$ vertices intersects the closed neighborhood of some vertex of C in at least $k + 1$ elements, implying $L_k(G) \leqslant k + 1$.
- Suppose $G = (S \hookleftarrow W, C, R)$. When $k = |S| - 1$ or $k = |S|$, it is straightforward that we can choose $B' = \emptyset$ or $B' = \{w_i\}$ for some $i \in \{1, 2\}$, respectively, and B is a maximum k-limited packing of G. Otherwise, by choosing $B' := \{w_i\} \cup A$ and $A \subseteq R \cup C$ with $|A| = k - |S|$, B is a k-limited packing of G; clearly $|N_G[v] \cap B| \leqslant k$ for each $v \in R \cup (S \hookleftarrow W)$ and $|N_G[v] \cap B| = |N_G[v] \cap (S \hookleftarrow W)| + |N_G[v] \cap A| \leqslant |S| + |A| = k$ for each $v \in C$. Therefore, $L_k(G) \geqslant |B| = k + 1$. In order to see the other inequality, let us observe that there is no k-limited packing of G with $k + 2$ vertices or more. Indeed, for every set B on $k + 2$ vertices, there exists $q \in C$ with $|B \cap N_G[q]| \geqslant k + 1$, i.e. B is not a k-limited packing of G. Then $L_k(G) \leqslant k + 1$. □

Concerning domination problems, recall that, for any given graph G, $\gamma_k(G) = +\infty$ for $k \geqslant \delta(G) + 2$. Hence, when G is a quasi-spider constructed from a spider H with partition (S, C, R), $\gamma_k(G)$ is known when H is thin and $k \geqslant 3$ (since $\delta(G) = 1$), and also when H is thick and $k \geqslant |S| + 1$ (since $\delta(G) = |S| - 1$). The following proposition considers the remaining values for k.

Proposition 4. *Let $H = (S, C, R)$, G be a quasi-spider obtained from H and $1 \leqslant k \leqslant \delta(G) + 1$.*

1. *If H is thin, then $\gamma_1(G) = |S|$ and*

$$\gamma_2(G) = \begin{cases} 2|S| + 1 & \text{when } G = (S \hookleftarrow S_2, C, R), \\ 2|S| & \text{otherwise}. \end{cases}$$

2. *If H is thick, then $\gamma_k(G) = k + 1$ for $1 \leqslant k \leqslant |S| - 1$ and*

$$\gamma_{|S|}(G) = \begin{cases} |S| + 2 & \text{when } G = (S, C \hookleftarrow W, R), \\ 2|S| & \text{when } G = (S \hookleftarrow K_2, C, R) \\ & \text{or } G = H, \\ 2|S| + 1 & \text{when } G = (S \hookleftarrow S_2, C, R). \end{cases}$$

Proof. 1. Suppose H is thin, $G = H$ and D is a k-tuple dominating set of G. Let $k = 1$. Each $x \in S$, itself or its neighbor must be in D, implying that $\gamma_1(G) \geqslant |S|$. On the other hand, C is a 1-tuple dominating set of G, thus $\gamma_1(G) = |S|$. Now let $k = 2$. Each vertex in S and its neighbor are in D, implying that $C \cup S \subseteq D$ and then $\gamma_2(G) \geqslant 2|S|$. Since $C \cup S$ is a 2-tuple dominating set of G, $\gamma_2(G) = 2|S|$.

The proofs of the statements for quasi-spiders that are not spiders easily follow.

2. Suppose H is thick and consider the following values for k:

- Take $1 \leqslant k \leqslant |S| - 1$. Every subset of C with $k + 1$ elements is a k-tuple dominating set of G, implying $\gamma_k(G) \leqslant k + 1$. On the other hand, given $D \subseteq C \cup S$ with $|D| = k$, there exists a vertex in $C \cup S$ whose closed neighborhood has at most $k - 1$ vertices in D, i.e., every k-tuple dominating set of G must have at least $k + 1$ vertices in $C \cup S$, implying $\gamma_k(G) \geqslant k + 1$.
- $k = |S|$. When $G = (S, C \hookleftarrow W, R)$, and—w.l.o.g—$c_1 \in C$ is replaced by W, $|N_G[s_1]| = |S|$ and this implies that $((C \hookleftarrow W) \setminus W) \cup \{s_1\} \subseteq D$ for every k-tuple dominating set of G. Moreover, since $|N_G[s] \cap (C \hookleftarrow W) \setminus W| = |S| - 1$ for $s \in S \setminus \{s_1\}$, every k-tuple dominating set of G must contain at least one vertex of W, say w_1 and also, s or w_2; thus $\gamma_k(G) \geqslant |S| + 2$. Besides, observe that every $x \in S \setminus \{s_1\}$ has exactly $|S|$ adjacent vertices in $C \hookleftarrow W$. Since $(C \hookleftarrow W) \cup \{s_1\}$ is a k-tuple dominating set of G, $\gamma_k(G) \leqslant |S| + 2$. Thus the result follows.
  When $G = H$, every vertex in S has exactly $|S|$ vertices in its closed neighborhood, then every k-tuple dominating set of G must contain the set $C \cup S$. Hence, $\gamma_k(G) \geqslant 2|S|$. But $C \cup S$ is a k-tuple dominating set of G, therefore $\gamma_k(G) = 2|S|$. In case $G = (S \hookleftarrow K_2, C, R)$, if $s_1 \in S$ is the vertex replaced by a K_2, the same argument is valid since every vertex in $S - \{s_1\}$ has exactly $|S|$ vertices in its closed neighborhood. The result follows.
  When $G = (S \hookleftarrow S_2, C, R)$, we have $|N_G[s]| = |S|$ for each $s \in S \hookleftarrow S_2$. Then $N_G[(S \hookleftarrow S_2)] \subseteq D$ for every k-tuple dominating set of G. This implies that $\gamma_{|S|}(G) \geqslant |S| + |C| + 1 = 2|C| + 1$. But $C \cup (S \hookleftarrow W)$ is a k-tuple dominating set of G and the result follows. □

As a corollary of the above two propositions, we have:

Corollary 5. *The k-Limited Packing and the k-Tuple Domination problems can be solved in polynomial time in quasi-spiders.*

We conclude this section by pointing out that spiders with empty head and those with partition $(S \hookleftarrow S_2, C, \emptyset)$ or $(S, C \hookleftarrow K_2, \emptyset)$ are split graphs, thus, they constitute a subclass of split graphs where both problems are polynomial time solvable.

4. The k-Limited Packing and the k-Tuple Domination problems in P_4-tidy graphs

P_4-tidy graphs were introduced by Rusu (see [7]), generalizing cographs and P_4-sparse graphs [11].

A *partner* of a path on four vertices P in G is a vertex $v \in V(G) - V(P)$ such that the subgraph induced by $V(P) \cup \{v\}$ has at least two paths on four vertices. A graph G is *P_4-tidy* if every path on four vertices has at most one partner. It is not difficult to prove that the family of P_4-tidy graphs is hereditary and self-complementary (for details see [7,11,14]). Quasi-spiders are involved in a known characterization of P_4-tidy graphs presented in [7].

Let us recall some well known facts that are valid for any graph. Given a graph G, if G or the complementary graph of G is not connected, G is the union or the join of two smaller graphs, respectively. This "decomposition" can be performed iteratively on each connected component of G or the complementary graph of G, until all subgraphs obtained, and also their complementary graphs, are connected. The complete decomposition can be performed in linear time and, once such a decomposition is obtained, we can reconstruct the given graph G in at most $O(\log_2 n)$ union and join operations, where n is the number of vertices in G. For details, see for instance [8].

The above arguments lead us to analyze the behavior of the packing and domination parameters under the union and join of graphs. We have:

Proposition 6. *Let G_1 and G_2 be two graphs and k a nonnegative integer number. Then,*

1. $L_k(G_1 \cup G_2) = L_k(G_1) + L_k(G_2)$ *and* $\gamma_k(G_1 \cup G_2) = \gamma_k(G_1) + \gamma_k(G_2)$,
2. $L_k(G_1 \vee G_2) = \max\{s + r\colon\ s, r \leqslant k,\ s, r \in \mathbf{Z}^+,\ s \leqslant L_{k-r}(G_1), r \leqslant L_{k-s}(G_2)\}$ *and*
3. $\gamma_k(G_1 \vee G_2) = \min\{s + r\colon\ s, r \leqslant k,\ s, r \in \mathbf{Z}^+,\ \gamma_{k-r}(G_1) \leqslant s \leqslant |V(G_1)|,\ \gamma_{k-s}(G_2) \leqslant r \leqslant |V(G_2)|\}$.

Proof. 1. It is straightforward.

2. Let B be a k-limited packing of $G_1 \vee G_2$ with $|B| = r + s$, where $s = |B \cap V(G_1)|$ and $r = |B \cap V(G_2)|$. For $x_1 \in V(G_1)$, $|N_G[x_1] \cap B| = r + |N_{G_1}[x_1] \cap B| \leqslant k$, implying that $B \cap V(G_1)$ is a $(k - r)$-limited packing of G_1 of size s. Thus $s \leqslant L_{k-r}(G_1)$. Similarly, $r \leqslant L_{k-s}(G_2)$. This proves that $L_k(G_1 \vee G_2) \leqslant \max\{s + r\colon\ s, r \leqslant k, s, r \in \mathbf{Z}^+,\ s \leqslant L_{k-r}(G_1),\ r \leqslant L_{k-s}(G_2)\}$.

Now take a $(k - r)$-limited packing B_1 of G_1 with $|B_1| = s$ and a $(k - s)$-limited packing B_2 of G_2 with $|B_2| = r$. It is not difficult to see that $B_1 \cup B_2$ is a k-limited packing of $G_1 \vee G_2$. Therefore $\max\{s + r\colon\ s, r \leqslant k,\ s, r \in \mathbf{Z}^+,\ s \leqslant L_{k-r}(G_1),\ r \leqslant L_{k-s}(G_2)\} \leqslant L_k(G_1 \vee G_2)$.

3. It is similar to the proof of item (2). □

Corollary 7. *For a given nonnegative integer k, graphs G_1 and G_2, and provided that $L_r(G_i)$ $(\gamma_r(G_i))$ can be computed in polynomial time for each integer number $r \leqslant k$ and $i = 1, 2$, $L_k(G_1 \cup G_2)$ $(\gamma_k(G_1 \cup G_2))$ and $L_k(G_1 \vee G_2)$ $(\gamma_k(G_1 \vee G_2))$ can be computed in polynomial time.*

Going back to P_4-tidy graphs, it is known that the complementary graph of a nontrivial connected P_4-tidy graph is also connected if and only if it is P_5, $\overline{P}_5$, C_5 or a quasi-spider. Then, Propositions 3 and 4 together with the results in this section allow us to state the following theorem:

Theorem 8. *The Limited Packing and the Multiple Domination problems are polynomial time solvable for P_4-tidy graphs.*

With regard to a question raised by Liao and Chang in [18] concerning the existence of another class of perfect graphs where the k-Tuple Domination problem is polynomial time solvable, we would like to add that Theorem 8 implies that the problems treated in this work are polynomial time solvable on the class of P_4-lite (C_5-free P_4-tidy) graphs. P_4-lite graphs were defined by Jamison and Olariu [15,16] and proved to be perfect by Hayward [10].

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