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A characterization of edge-perfect graphs and the complexity of recognizing some combinatorial optimization games

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ABSTRACT

We characterize edge-perfect graphs and prove that it is *co-NP*-complete to recognize them. In consequence, recognizing the defining matrices of totally balanced packing games is also *co-NP*-complete, in contrast with the polynomiality for the covering case. In addition, we solve the computational complexity of universally balanced (with respect to the resources constraints) packing games.

1. Introduction

The notion of edge-perfection in graphs considered in this work was introduced in [1] when reformulating an open problem in game theory. Other concepts of edge-perfection in graphs can be found in [2,3].

Packing and covering games were defined by Deng et al. in [4] as particular classes of (cooperative) combinatorial optimization games.

A cooperative game (N, v) consists of a set of *players* N and a *value function* $v : 2^N \to \mathbb{R}_+$. Given (N, v) and $R \subseteq N$, (R, v_R) is the *subgame* induced by R if $v_R(K) = v(K)$, for each subset K of R.

Given a 0, 1 matrix $A = (a_{ij})$ of order $n \times m$ without zero columns, if $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, (N, v_A^p) is the *(simple) packing game defined by* A and (M, v_A^c) is the *(simple) covering game defined by* A if

- for $S \subseteq N$, $v_A^p(S) = \max\left\{\mathbf{1}^t x : Ax \le e_S,\ x \in \{0, 1\}^m\right\}$;
- for $T \subseteq M$, $v_A^c(T) = \min\left\{\mathbf{1}^t y : y^t A \ge e_T^t,\ y \in \{0, 1\}^n\right\}$;

where $\mathbf{1}$ is the vector of all ones and $e_R \in \{0, 1\}^k$ is the characteristic vector of $R \subseteq \{1, \ldots, k\}$ [4].

For $S \subseteq N$ and $T \subseteq M$, let us denote by $A_{S,T}$, the submatrix of A with rows indexed by the elements in S and columns, by the elements in T. The induced subgames preserve the combinatorial structure of the original game. More precisely, if for each $S \subseteq N$ we consider

$$T(S) = \left\{j \in M : a_{ij} = 0, \forall i \in N \setminus S\right\},$$

it is not difficult to see that the subgame of (N, v_A^p) induced by $S \subseteq N$ is the packing game $(S, v_{A_{S,T(S)}}^p)$. Similarly, the subgame of (M, v_A^c) induced by $T \subseteq M$ is the covering game $(T, v_{A_{N,T}}^c)$.

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In [4] it is proved that a 0, 1 matrix A defines a *balanced* covering (resp. packing) game if and only if the problem $\min\{\mathbf{1}^t y : y^t A \geq e_T^t,\ y \geq \mathbf{0}\}$ (resp. $\max\{\mathbf{1}^t x : Ax \leq e_S,\ x \geq \mathbf{0}\}$) has an integer optimal solution. For the general definition of game balancedness, see for example [5].

Deng et al. in [6] proved that if the covering game defined by a general 0, 1 matrix A is *totally balanced* – all induced subgames are balanced – then the packing game defined by A is totally balanced as well. Conversely, they proved that if A defines a totally balanced packing game, then the covering game defined by A is just balanced. The problems of finding a complete characterization for totally balanced covering and totally balanced packing game defining matrices were left open.

In the rest of this section we will say that A is TBP (TBC) when A defines a totally balanced packing (covering) game.

In 2005, van Velzen [7] solved the problem for the covering case, showing that the only TBC matrices are *perfect* matrices, i.e., clique-node matrices of perfect graphs. It is well known that both, clique-node matrices and perfect graphs, can be recognized in polynomial time (see for example [8, p. 24] and [9], respectively).

In contrast with the covering case, a complete characterization for TBP matrices as well as a study of the computational complexity of the corresponding recognition problem remain open.

Motivated by van Velzen's result, Escalante et al. [1] proposed a reformulation of the packing case in terms of graphs. First, they showed that A is TBP if and only if the covering and packing games defined by $A_{S,T(S)}$ are balanced, for each $S \subseteq N$. Besides, when A has exactly two ones per row, i.e. it is the edge-node incidence matrix of a graph $G(A)$, the packing and covering games defined by A are balanced if and only if the stability and edge covering numbers of $G(A)$ coincide. In this case, the graph is said to be *edge-good*. Finally, they proved that there is a 1–1 correspondence between – packing subgame defining – matrices of the form $A_{S,T(S)}$ for $S \subseteq N$, and subgraphs obtained from $G(A)$ by deleting the endpoints of a subset of its edge set. These subgraphs are called *edge-subgraphs*, and a graph is called *edge-perfect* if all its edge-subgraphs are edge-good. In this way, the authors arrive at the following theorem which motivates the present work:

Theorem 1 (*[1]*)**.** *Let A be a 0, 1 matrix with two ones per row and let G be the graph such that A is its edge-node incidence matrix. Then A is TBP if and only if G is edge-perfect.*

Recently, the notion of *universally balanced packing games* was introduced [10]. In particular, the authors showed that universally balanced (with respect to the objective) packing games can be recognized in polynomial time. In contrast, the computational complexity of recognizing universally balanced (with respect to the resources constraints) packing games is still unknown. However, the authors remarked the following:

Remark 2. The property of universally balanced with respect to the resources constraints implies that the core of every subgame of the game is non-empty, i.e. that the game is totally balanced.

In this work, we present a characterization of edge-perfect graphs which allows us to prove that the corresponding recognition problem is *co-NP*-complete. As a consequence, we obtain the computational complexity of recognizing 0, 1 TBP matrices and recognizing universally balanced (with respect to the resources constraints) packing games.

Besides, we attain as by-products the *NP*-completeness of two problems in graphs: the "undirected" and "odd cycle" versions of the known *NP*-complete problem *Path with Forbidden Pairs*.

A preliminary version of some results in this paper without complete proofs can be found in [11,12].

2. Definitions and notation

Throughout this work graphs are simple, that is, they have neither parallel edges nor loops. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The edge with endpoints $u, v \in V(G)$ is indicated by uv. Given $s, t \in V(G)$, an *st-path* in G is a simple path with extremes s and t. The *edge-node incidence* matrix of G is the 0, 1 matrix with columns indexed by $V(G)$ and whose rows are the incidence vectors of its edges. When $E' \subseteq E(G)$, $V(E')$ denotes the set of the endpoints of the edges in E'. For $v \in V(G)$, $N_G(v)$ denotes the *neighborhood* of v in G, i.e., $N_G(v) = \{w \in V(G) : vw \in E(G)\}$. The *degree* in G of vertex v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. Two vertices $v, w \in V(G)$ are called *twins* if $N_G(v) = N_G(w)$. In this case, we say that v (w) has a twin in G.

Given $T \subseteq V(G)$, $G \setminus T$ denotes the subgraph of G obtained by *deleting* the vertices in T, i.e., the subgraph with vertex set $V(G) \setminus T$ and edge set $E(G) \setminus \{vw : v \in T,\ w \in V(G)\}$. An *(induced) subgraph* of G is a graph obtained from G by deleting a subset of vertices.

When it is not necessary to name the subset of deleted vertices, we simply use the notation $G' \subseteq G$ for a subgraph G' of G. A subgraph G' of G is an *edge-subgraph* – denoted by $G' \subseteq_{\mathcal{E}} G$ – if there exists $E' \subseteq E(G)$ such that $G' = G \setminus V(E')$. When $G' \subseteq_{\mathcal{E}} G$ and $G' \neq G$, we say that G' is a *proper* edge-subgraph.

A cycle on r vertices is denoted by C_r; C_3 is also called a *triangle*. An *odd chordless cycle* in G is a subgraph of G that is C_{2k+1}, for some $k \geq 1$.

A *diamond* is a complete graph on four vertices minus one edge.

A *stable set* in G is a set of pairwise nonadjacent vertices. The *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum sized stable set in G. An *edge cover* of G is a subset of $E(G)$ such that every vertex in G is an endpoint of some edge in the cover. When G is connected and has at least two vertices, the *edge covering number* of G, denoted by $\rho(G)$, is the cardinality of a minimum sized edge cover in G. When $|V(G)| = 1$, we define $\rho(G) = 1$ and when G is not connected, $\rho(G)$ is the sum of the edge covering numbers of its connected components. Let us mention that edge-good graphs are also known as Kőnig–Egervary graphs.

It is clear that $\alpha(G) \leq \rho(G)$, for every G. A graph G is *edge-good* if $\alpha(G) = \rho(G)$. At this point, we are able to present the concept of edge-perfection that we deal with in this work.

Definition 3 ([1]). A graph G is *edge-perfect* if, for every $G' \subseteq_{\mathcal{E}} G$, G' is edge-good, i.e., $\alpha(G') = \rho(G')$.

From Kőnig's edge cover theorem [13], bipartite graphs are edge-good. Since each subgraph of a bipartite graph is also bipartite, bipartite graphs are edge-perfect. On the other hand, odd cycles and complete graphs with at least three vertices are not edge-good.

3. Characterizations of edge-perfect graphs

The following property gives a characterization of those subgraphs which are not edge-subgraphs.

Property 4. *Let G be any graph and $G' \subseteq G$. Then G' is not an edge-subgraph if and only if there exists $v \in V(G) \setminus V(G')$ with $N_G(v) \subseteq V(G')$.*

The key is that we cannot get rid of v by removing the endpoints of some edges from G without removing vertices from G' at the same time. Conversely, if for each $v \notin V(G')$ there is an edge vw_v with $w_v \notin V(G')$, G' is an edge-subgraph because it can be obtained by deleting the endpoints of the edges vw_v, with $v \notin V(G')$.

We introduce the following concept:

Definition 5. Given $G' \subseteq G$ and $v \in V(G)$, v is a *guarantor* of G' if $v \notin V(G')$ and $N_G(v) \subseteq V(G')$.

In the way of providing a characterization for edge-perfect graphs, we will frequently make use of the following lemma:

Lemma 6. *Let $G' \subseteq G$, w be a guarantor of G' and $G'' \subseteq G$ with $w \in V(G'') \subseteq V(G') \cup \{w\}$. Then, every guarantor w' of G'' with $w' \notin V(G')$ is also a guarantor of G'.*

Proof. Let w' be a guarantor of G''. Then $N_G(w') \subseteq V(G'') \subseteq V(G') \cup \{w\}$. In order to demonstrate that w' is also a guarantor of G', it is enough to show that w is not adjacent to w'. Since w is a guarantor of G', $N_G(w) \subseteq V(G')$. Consequently, $w' \notin N_G(w)$ because $w' \notin V(G')$. □

The following result provides a necessary condition for edge-perfect graphs:

Lemma 7. *Let G be an edge-perfect graph and let $C \subseteq G$ be a triangle without guarantors of degree one. Then, C has a guarantor of degree two and every guarantor of C of degree two has a twin in G.*

Proof. Since G is edge-perfect and C is not edge-good, following Property 4, C has a guarantor. By hypothesis, C has no guarantors of degree one, so all guarantors of C have degree two or three. We will prove that C has a guarantor of degree two by proving that, if C has a guarantor of degree three, it also has one of degree at most two.

Let w be a guarantor of C of degree three. Notice that, in this case, $d_G(v) \geq 3$ for every $v \in V(C)$. We will consider separately the cases $d_G(v) = 3$ for some $v \in V(C)$ and $d_G(v) \geq 4$ for all $v \in V(C)$.

Case 1. Let $v \in V(C)$ such that $d_G(v) = 3$. We apply Lemma 6 with $G' = C$ and G'' the complete subgraph induced by $V(C) \cup \{w\}$. Since G'' is not edge-good, G'' has a guarantor w'. Since $w' \notin V(C)$, by Lemma 6 w' is a guarantor of C, thus $N_G(w') \subseteq V(C)$. Moreover, seeing that $d_G(v) = 3$, w' is not adjacent to v, which implies that $d_G(w') \leq 2$.

Case 2. Suppose $d_G(v) \geq 4$ for every $v \in V(C)$. We will again apply Lemma 6 where $G' = C$, G'' is the triangle induced by w and any two vertices in $V(C)$ and w' is a guarantor of G''. It is clear that $w' \notin V(C)$ because the vertex in $V(C) \setminus V(G'')$ has degree at least four. By Lemma 6, it follows that w' is also a guarantor of C. Since $N_G(w') \subseteq V(G') \cap V(G'')$, we have $d_G(w') \leq 2$.

To complete the proof, we will prove that every guarantor of C of degree two has a twin in G.

Let w be any guarantor of C with $d_G(w) = 2$ and let v be the vertex in $V(C) \setminus N_G(w)$. If $d_G(v) = 2$, we are done; otherwise, consider $G' = C$, G'', the triangle induced by $N_G(w) \cup \{w\}$ and w', a guarantor of G''. Notice that $w' \neq v$ because $d_G(v) \geq 3$ and w is not adjacent to v ($d_G(w) = 2$). Thus $w' \notin V(C)$. Lemma 6 implies that w' is a guarantor of C and $d_G(w') \leq 2$. Since C has no guarantor of degree one, it holds that $d_G(w') = 2$, implying that w' is a twin of w. □

The previous lemma can be partially extended to any odd chordless cycle in an edge-perfect graph.

Proposition 8. *Let G be an edge-perfect graph. Then, every odd chordless cycle in G without guarantors of degree one has a guarantor of degree two, having a twin.*

Proof. We proceed by induction on k, where $2k + 1$ is the number of vertices of an odd chordless cycle. The case $k = 1$ was proved in Lemma 7.

Assume that the t holds for every odd chordless cycle in G with at most $2k + 1$ vertices and take C an odd chordless cycle in G with $2k + 3$ vertices and without guarantors of degree one.

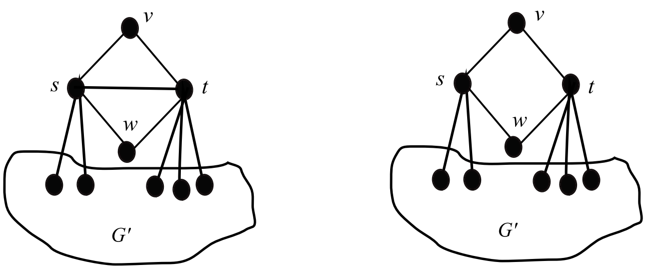

Fig. 1. A scheme of graphs with a two-twin pair $\{v, w\}$.

Let us first consider the case where some guarantor w of C has two neighbors v_1, v_2 at distance in C different from two. When v_1 and v_2 are adjacent, let C' be the triangle with $V(C') = \{w, v_1, v_2\}$; when they are not, let C' be the subgraph induced by w and the vertices in the odd v_1v_2-path P in C. Observe that, in the last case, P has length at most $(2k+3) - 4 = 2k - 1$ implying that $|V(C')| \leq 2k + 1$. From the construction of C', no vertex of $V(C)$ can be a guarantor of C', thus Lemma 6 implies that every guarantor of C' is a guarantor of C. By the induction hypothesis, C' has a guarantor w of degree two with a twin. Then, w is a guarantor of C of degree two with a twin.

Next, let us observe that, if some guarantor of C has degree at least three, two of its neighbors are at a distance in C different from two, and from the previous analysis the statement holds. It only remains to consider the case where every guarantor of C has exactly two neighbors and they are at a distance two in C. Let us denote by S the set of guarantors of C and by w_C the vertex in $V(C)$ with $N_C(w_C) = N_G(w)$, for each $w \in S$.

If $d_G(w_C) = 2$ for some $w \in S$, it turns out that w_C and w are twins and the statement holds.

Assume that $d_G(w_C) \geq 3$ for all $w \in S$ and consider the set

$$\mathcal{G} = \{C' \text{ odd chordless cycle in } G : \ V(C') \subseteq V(C) \cup S \text{ and } C' \neq C\}.$$

Let us point out the following fact about any cycle C' in $\mathcal{G}$:

(1) for every $v \in V(C) \setminus V(C')$, $v = w_C$ for some $w \in S \cap V(C')$ and then, v has degree at least three.

Let us observe that no vertex in $V(C)$ can be a guarantor of C'. Indeed, from (1), if $w \in V(C)$ is a guarantor of C' then $d_G(w) \geq 3$ and, since C is chordless, w is adjacent to a vertex $w' \in S \cap V(C')$. Besides, since every vertex in S has degree two, $N_G(w') \subseteq V(C')$, contradicting that $w \notin V(C')$.

Moreover, no guarantor w of C' can be adjacent to a vertex in S, thus $N_G(w) \subseteq V(C)$.

We conclude that every guarantor of $C' \in \mathcal{G}$ is a guarantor of C.

Now, take $C^* \in \mathcal{G}$ such that $|V(C^*) \cap S| = \max\{|V(C') \cap S| : C' \in \mathcal{G}\}$ and w^* a guarantor of C^*.

In particular, we have already showed that $w^* \in S$. If $w^*_C \in V(C^*)$, then $V(C^*) \setminus \{w^*_C\} \cup \{w^*\}$ induces an odd chordless cycle $C'' \in \mathcal{G}$ with $|V(C'') \cap S| = |V(C^*) \cap S| + 1$, contradicting the definition of C^*. Therefore, $w^*_C \notin V(C^*)$. From observation (1), $w^*_C = w_C$ for some $w \in S \cap V(C^*)$ which implies that w^* and w are twins and the statement holds. □

The previous result motivates us to introduce new terminology. Given a graph G, a *two-twin pair* in G is a pair of twins of degree two (see Fig. 1).

Besides, given an odd chordless cycle $C \subseteq G$, v is a 1-*guarantor* of C if v is guarantor of C and $d_G(v) = 1$, and it is a 2T-*guarantor* of C if v is a guarantor of C, $d_G(v) = 2$ and v has a twin. Observe that if v is a 2T-guarantor of C and w is a twin of v, then $\{v, w\}$ is a two-twin pair.

We will now show that the condition in Proposition 8 is also sufficient:

Theorem 9. *A graph G is edge-perfect if and only if for every odd chordless cycle C in G, C has a* 1*-guarantor or a* 2T*-guarantor.*

Proof. The "only if part" corresponds to Proposition 8. We will prove the "if part" by induction on $n = |V(G)|$.

The result holds for $n \leq 3$. Suppose that the result holds for all graphs with at most n vertices and take a graph G with $|V(G)| = n + 1$ such that every odd chordless cycle in G has a 1-guarantor or a 2T-guarantor.

Given $G' \subseteq_e G$ and C an odd chordless cycle in G', from the hypothesis C has a 1-guarantor or a 2T-guarantor $w \in V(G)$. In fact, $w \in V(G')$; if not, to obtain G' from G we should delete w together with some of its neighbors, but they are all part of G' since $N_G(w) \subseteq V(C) \subseteq V(G')$. Then, every odd chordless cycle in G' has a 1-guarantor or a 2T-guarantor in $V(G')$.

From the previous analysis and the induction hypothesis, it follows that every proper edge-subgraph of G is edge-perfect. To conclude that G is edge-perfect, it only remains to prove that G is edge-good, i.e. that $\alpha(G) = \rho(G)$.

If G has a vertex v with $d_G(v) = 1$ and $N_G(v) = \{w\}$, vw belongs to every minimum edge cover of G and every maximum stable set in G contains v or w. Then, by considering $G' = G \setminus \{v, w\}$, we have $\rho(G) = \rho(G') + 1$ and $\alpha(G) = \alpha(G') + 1$. Since G' is a proper edge-subgraph, G' is edge-perfect and $\alpha(G') = \rho(G')$. Then, G is edge-good.

Otherwise – $d_G(v) \geq 2$, for all v – there is an odd chordless cycle in G having a 2T-guarantor v. Let $\{s, t, w\} \in V(G)$ with $N_G(v) = N_G(w) = \{s, t\}$ and $G' = G \setminus \{s, t, v, w\}$. We will demonstrate that $\alpha(G) = \alpha(G') + 2$ and $\rho(G) = \rho(G') + 2$. Since the subgraph induced by $\{s, t, v, w\}$ (see Fig. 1) has stability and edge covering numbers equal to two, we have $\alpha(G) \leq \alpha(G') + 2$

and $\rho(G) \geq \rho(G') + 2$. On the other hand, let S and D be a maximum stable set and a minimum edge cover of G' respectively. Then $S \cup \{v, w\}$ and $R \cup \{sw, vt\}$ are a stable set and an edge cover of G, respectively. Therefore, $\alpha(G) \geq \alpha(G') + 2$ and $\rho(G) \leq \rho(G') + 2$. Again $\alpha(G') = \rho(G')$ because G' is a proper edge-subgraph of G, thus G is edge-good. □

We end this section by giving a simple characterization of edge-perfect claw-free graphs, derived from Theorem 9:

Corollary 10. *Let G be a connected edge-perfect claw-free graph without pendant vertices. Then G is bipartite or G is a diamond.*

Proof. We have to show that if G is non bipartite, then G is a diamond.

Let C be an odd chordless cycle in G. Theorem 9 implies that there exists w, a 2T-guarantor of C. Let $\{s, t, v\} \subseteq V(G)$ with $N_G(v) = N_G(w) = \{s, t\} \subseteq V(C)$.

If C has at least five vertices or $v \notin V(C)$, by considering $s' \in N_C(s)$ such that $s' \neq t$ and $s' \neq v$, we have $\{v, w, s', s\}$ induces a claw in G, a contradiction. Thus C is a triangle, $v \in V(C)$ and $V(C) \cup \{w\}$ induces a diamond. It only remains to prove that $V(G) = V(C) \cup \{w\}$. If there exists $v' \in V(G) \setminus (V(C) \cup \{w\})$, v' is adjacent to at least one neighbor of w, because G is connected and $d_G(w) = d_G(v) = 2$. W.l.o.g, let us assume that v' is adjacent to s. Therefore, $\{v', s, v, w\}$ induces a claw in G, a contradiction. □

4. Computational complexity results

In this section we focus on the computational complexity of the *Edge-Perfect* problem (EP), formally defined as:

INSTANCE: G, a graph.

QUESTION: Is G edge-perfect?

From Theorem 9, we derive that EP is in *co-NP* and that it is solvable in polynomial time on graphs for which the set of odd chordless cycles can be polynomially computed.

For example, EP is solvable in polynomial time on perfect graphs since each odd chordless cycle is a triangle. Also, on *outerplanar* graphs (graphs having a planar drawing in which all of the vertices belong to the unbounded face) since the number of chordless cycles is bounded by the number of vertices.

In addition, from Corollary 10 we also derive that edge-perfection can be polynomially recognized on claw-free graphs. In fact, given a claw-free graph G, by deleting from G every pendant vertex together with their neighbors, the resulting graph is claw-free and moreover, it is edge-perfect if and only if G is. By repeating this procedure, we can polynomially obtain a claw-free graph $\tilde{G}$ without pendant vertices which is edge-perfect if and only if G is. In order to decide if $\tilde{G}$ is edge-perfect, we only need to verify if each connected component of $\tilde{G}$ is bipartite or a diamond.

Next, we show that there exists an *NP*-complete problem which can be polynomially reduced to the complement of EP. With this purpose in mind, we first reformulate Theorem 9.

Given a graph G, we call $\mathcal{N}(G)$ the set of pairs from $V(G)$ given by the neighbors of the two-twin pairs in G. If G has not got pendant vertices, from Theorem 9 we know that G is edge-perfect if and only if, for every odd chordless cycle $C \subseteq G$, some pair of $\mathcal{N}(G)$ is contained in $V(C)$. Thus, Theorem 9 can be reformulated in the following way:

Theorem 11. *Let G be a graph without pendant vertices. Then, G is not edge-perfect if and only if there exists an odd chordless cycle in G containing at most one vertex from each pair in $\mathcal{N}(G)$.*

This reformulation of Theorem 9 leads us to get closer to the known *NP*-complete problem *Path with Forbidden Pairs* (DPFP) on a digraph. DPFP is formulated as follows (see problem [GT54], p. 203 in [14]):

INSTANCE: $D = (N, A)$, a directed graph; $s, t \in N$; $\mathcal{T}$, a collection of pairs from N.

QUESTION: Is there a directed path from s to t in D containing at most one vertex from each pair in $\mathcal{T}$?

Given $D, s, t, \mathcal{T}$ defining an instance of DPFP, a directed path from s to t in D containing at most one vertex from each pair in $\mathcal{T}$ will be called an *admissible path in D*.

We will reduce polynomially DPFP to the complement of EP. To make the reduction clearer, we will go through three intermediate new "forbidden pairs problems".

We first consider the "undirected version" of DPFP. Formally, the problem called *Undirected Path with Forbidden Pairs* (UPFP) is defined as:

INSTANCE: G, a graph; $\alpha, \beta \in V(G)$; $\mathcal{P}$, a collection of pairs from $V(G)$.

QUESTION: Is there an $\alpha\beta$-path in G containing at most one vertex from each pair in $\mathcal{P}$?

Given $G, \alpha, \beta, \mathcal{P}$ defining an instance of UPFP, an *admissible path in G* is an $\alpha\beta$-path in G containing at most one vertex from each pair in $\mathcal{P}$.

We have:

Theorem 12. *UPFP is NP-complete.*

Proof. Clearly, UPFP is in *NP*. We will reduce polynomially DPFP to UPFP.

Let $D = (N, A)$, s, t and $\mathcal{T}$ define an instance of DPFP. The arc in A from node $u \in N$ to node $v \in N$ will be indicated by (u, v).

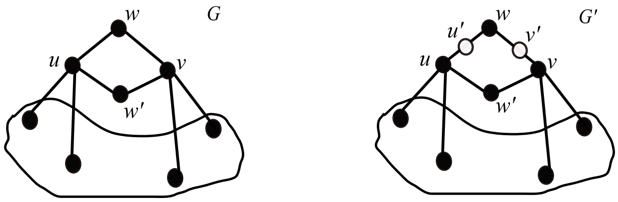

Fig. 2. $\{w, w'\}$ is a two-twin pair in G but not in G'.

By considering two "dummy" vertices s' and t', we construct an equivalent instance of DPFP given by $D' = (N', A'), s', t'$ and $\mathcal{T}$, where $N' := N \cup \{s', t'\}$ and $A' := A \cup \{(s', s), (t, t')\}$.

Now, let G be a graph where $V(G) = A'$ and

$$E(G) = \{(a, b)(c, d) : (a, b) \in V(G),\ (c, d) \in V(G) \text{ and } (a = d \text{ or } b = c)\}.$$

Consider also $\alpha = (s', s)$, $\beta = (t, t')$.

Given P a directed path from s' to t' in D', the vertices in G corresponding to the set of arcs of P together with α and β, induce an $\alpha\beta$-path P in G. Given an $\alpha\beta$-path in G, the set of arcs of D' that correspond to the vertices of P form a directed path from s' to t' in D if and only if for any two such arcs (a, b) and (c, d) it holds that $a \neq c$ and $b \neq d$.

Therefore, if we define $\mathcal{P}_1 := \{\{(a, b), (c, d)\} : b = d \text{ or } a = c\}$, there is a 1–1 correspondence between directed paths from s to t in D' and $\alpha\beta$-paths in G that meet each pair in $\mathcal{P}_1$ in at most one element.

Finally, by considering $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$ with

$$\mathcal{P}_2 := \{\{(a, b), (c, d)\} : \{a, c\} \in \mathcal{T} \text{ or } \{b, d\} \in \mathcal{T}\},$$

there is a 1–1 correspondence between admissible paths in D' and admissible paths in G. $\square$

Let us observe that, from the construction of the graph G in the proof of Theorem 12, UPFP remains *NP*-complete on instances satisfying $\alpha\beta \notin E(G)$ and $\{\alpha, \beta\} \notin \mathcal{P}$.

Let us now consider an "odd cycle version" of UPFP. The problem called *Odd Cycle with Forbidden Pairs* (OCFP) is defined as:

INSTANCE: G, a graph; $\mathcal{T}$, a collection of pairs from $V(G)$.

QUESTION: Is there an odd chordless cycle in G containing at most one vertex from each pair in $\mathcal{T}$?

Given an instance G, $\mathcal{T}$ of OCFP, an *admissible odd chordless cycle* in G is an odd chordless cycle in G that contains at most one vertex from each pair in $\mathcal{T}$.

We can state:

Theorem 13. *OCFP is NP-complete.*

Proof. Clearly, OCFP is in *NP*. We will reduce polynomially UPFP to OCFP.

Consider an instance of UPFP defined by a graph G, $\alpha, \beta \in V(G)$ with $\alpha\beta \notin E(G)$ and a collection of pairs $\mathcal{P}$ from $V(G)$, with $\{\alpha, \beta\} \notin \mathcal{P}$. Let

$$V' = \{x_e : e \in E(G)\} \quad \text{and} \quad E' = \bigcup_{e=uv \in E(G)} \{ux_e,\ x_e v\}.$$

Take now the instance of OCFP given by a graph G' with $V(G') = V(G) \cup V'$ and $E(G') = E' \cup \{\alpha\beta\}$, and a family $\mathcal{T}$ with $\mathcal{T} = \mathcal{P}$.

There is a 1–1 correspondence between $\alpha\beta$-paths in G and $\alpha\beta$-paths in G'. Moreover, since G' without the edge $\alpha\beta$ is bipartite, $\alpha\beta$ belongs to every odd chordless cycle in G'. Therefore, there is a 1–1 correspondence between $\alpha\beta$-paths and odd chordless cycles in G'. Since $\mathcal{T} = \mathcal{P}$, there exists an admissible path in G if and only if there exists an admissible odd chordless cycle in G'. $\square$

Let us observe that OCFP remains *NP*-complete for graphs without pendant vertices. Moreover, it remains *NP*-complete for graphs without two-twin pairs. Indeed, given G, $\mathcal{T}$ defining an instance of OCFP, if $\{w, w'\}$ is a two-twin pair in G with $N_G(w) = N_G(w') = \{u, v\}$, we define G' (see Fig. 2) with

$$V(G') = V(G) \cup \{u', v'\} \quad \text{and} \quad E(G') = E(G) \setminus \{wu, wv\} \cup \{wu', u'u, wv', v'v\}.$$

It turns out that instances defined by G and $\mathcal{T}$ and by G' and $\mathcal{T}$ are equivalent for OCFP, and the number of two-twin pairs in G' is one less than the number of two-twin pairs in G.

Finally, we have:

Theorem 14. *DPFP can be polynomially reduced to the complement of EP.*

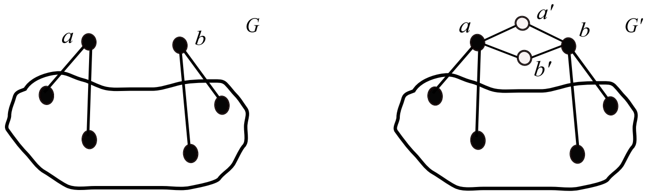

Fig. 3. Transformation in Theorem 14.

Proof. Taking into account Theorems 12 and 13, it is enough to polynomially reduce OCFP to the complement of EP. Let us consider an instance of OCFP given by G and $\mathcal{T}$, where G has neither two-twin pairs nor vertices of degree one. Let us define an instance of EP in the following way. First and for each pair $\{a, b\} \in \mathcal{T}$, add a two-twin pair $\{a', b'\}$ with neighborhood $\{a, b\}$ to G (see Fig. 3, where $\mathcal{T} = \{\{a, b\}\}$).

Formally, define the sets

$$V' = \bigcup_{\{a,b\}\in\mathcal{T}} \{a', b'\} \quad \text{and} \quad E' = \bigcup_{\{a,b\}\in\mathcal{T}} \{a'a, a'b, b'a, b'b\}$$

and consider the instance of EP given by the graph G' with $V(G') := V(G) \cup V'$ and $E(G') := E(G) \cup E'$. Under the notation of Theorem 11, we have $\mathcal{N}(G') = \mathcal{T}$.

Notice that every odd chordless cycle in G' which is not a cycle in G contains a pair in $\mathcal{N}(G')$. Therefore, there is an admissible odd chordless cycle in G if and only if there is an admissible odd chordless cycle in G'. From Theorem 11, there is an admissible odd chordless cycle in G' if and only if G is not edge-perfect. $\square$

As a corollary, we obtain the announced computational complexity of EP:

Theorem 15. *EP is co-NP-complete.*

Finally, by taking into account the result in Theorem 1, we answer the open question stated in [6]:

Theorem 16. *Recognizing if a* $0, 1$*-matrix defines a totally balanced packing game problem is co-NP-complete.*

This result together with Remark 2 allows us to derive the computational complexity of the type of balanced games recently treated in [10]:

Corollary 17. *Recognizing if a* $0, 1$*-matrix defines a universally balanced (with respect to resources constraints) packing game problem is NP-hard.*

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